



# Comprehensive Curriculum

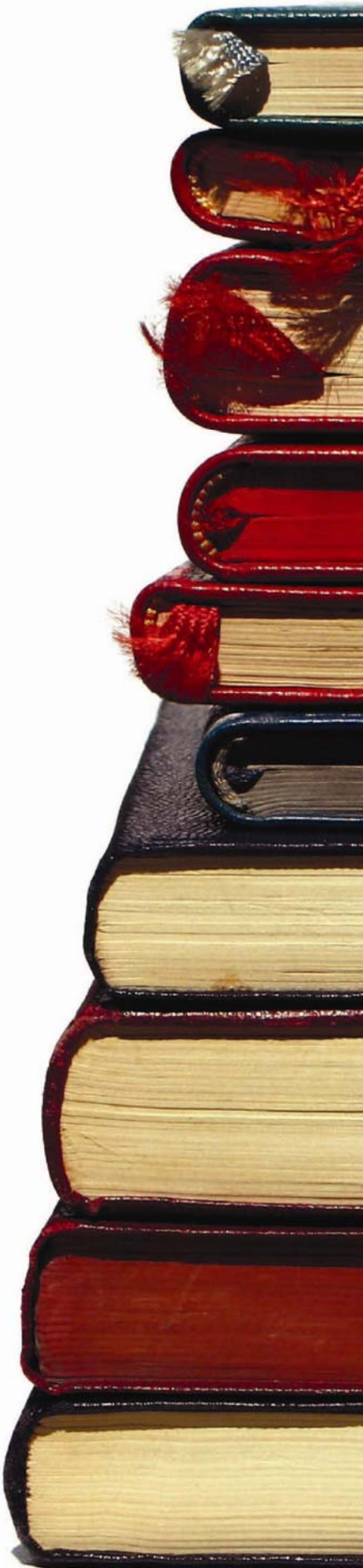
Revised 2008

## Advanced Mathematics Functions and Statistics (formerly Advanced Mathematics II)



Louisiana Department of  
**EDUCATION**

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**Advanced Mathematics – Functions and Statistics  
(formerly Advanced Mathematics II)**

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## *Louisiana Comprehensive Curriculum, Revised 2008* Course Introduction

The Louisiana Department of Education issued the *Comprehensive Curriculum* in 2005. The curriculum has been revised based on teacher feedback, an external review by a team of content experts from outside the state, and input from course writers. As in the first edition, the *Louisiana Comprehensive Curriculum*, revised 2008 is aligned with state content standards, as defined by Grade-Level Expectations (GLEs), and organized into coherent, time-bound units with sample activities and classroom assessments to guide teaching and learning. The order of the units ensures that all GLEs to be tested are addressed prior to the administration of *iLEAP* assessments.

### **District Implementation Guidelines**

Local districts are responsible for implementation and monitoring of the *Louisiana Comprehensive Curriculum* and have been delegated the responsibility to decide if

- units are to be taught in the order presented
- substitutions of equivalent activities are allowed
- GLEs can be adequately addressed using fewer activities than presented
- permitted changes are to be made at the district, school, or teacher level

Districts have been requested to inform teachers of decisions made.

### **Implementation of Activities in the Classroom**

*Incorporation of activities into lesson plans is critical to the successful implementation of the Louisiana Comprehensive Curriculum.* Lesson plans should be designed to introduce students to one or more of the activities, to provide background information and follow-up, and to prepare students for success in mastering the Grade-Level Expectations associated with the activities. Lesson plans should address individual needs of students and should include processes for re-teaching concepts or skills for students who need additional instruction. Appropriate accommodations must be made for students with disabilities.

### **New Features**

*Content Area Literacy Strategies* are an integral part of approximately one-third of the activities. Strategy names are italicized. The link ([view literacy strategy descriptions](#)) opens a document containing detailed descriptions and examples of the literacy strategies. This document can also be accessed directly at <http://www.louisianaschools.net/1de/uploads/11056.doc>.

A *Materials List* is provided for each activity and *Blackline Masters (BLMs)* are provided to assist in the delivery of activities or to assess student learning. A separate Blackline Master document is provided for each course.

The *Access Guide to the Comprehensive Curriculum* is an online database of suggested strategies, accommodations, assistive technology, and assessment options that may provide greater access to the curriculum activities. The *Access Guide* will be piloted during the 2008-2009 school year in Grades 4 and 8, with other grades to be added over time. Click on the *Access Guide* icon found on the first page of each unit or by going directly to the url <http://mconn.doe.state.la.us/accessguide/default.aspx>.



## Advanced Math – Functions and Statistics Unit 1: Functions

**Time Frame:** Approximately 4.5 weeks



### Unit Description

This unit will continue the study of functions begun in Algebra I. The major topics include a more rigorous study of the graphs of functions and the different ways to express functions (tables of values, algebraic equations, and verbal descriptions). Functions that model physical phenomena will be used as examples to study the concepts of domain, range, and function composition. Translations, dilations, and reflections of functions will also be explored in order to deepen student understanding of functions and the families to which they belong.

### Student Understandings

Students will deepen their understanding of the meaning of a function. They will examine more intricate functions and will relate the graphs of those functions to the previously studied concepts of independent variable, dependent variable, domain, and range. Students will understand not only how to produce combinations of functions and inverse functions, but also how these concepts relate to realistic situations. Students will produce and interpret mathematical models. By studying these models, students will understand that functions have specific shapes and can be continuous or discrete. They will consider both local and global behavior of functions and how these concepts affect how well their models fit.

### Guiding Questions

1. Can students read a graph and state important characteristics like domain, range, increasing, decreasing and constant intervals, local and global extrema (max and min),  $y$  values given  $x$  values,  $x$  values given  $y$  values, and intervals where  $f(x) > 0$  and  $f(x) < 0$ ?
2. Can students interpret graphs of functions as being the set of ordered pairs satisfying the condition  $y = f(x)$  and identify domains and ranges based on the description of the data, the graph of the function, and the function definition?
3. Can students find similarities and differences between the families of functions?
4. Can students write and graph translations, dilations, and reflections of functions?
5. Can students write and graph piecewise functions?
6. Can students produce, evaluate, and graph combinations of functions?
7. Can students identify when it is appropriate to express one variable as a function of another?
8. Can students find the inverse of a function?

9. Can students identify and name the properties of 1-to-1 functions?
10. Can students accurately use function notation to describe a real-world situation?
11. Can students show the relationships between graphs, tables of values, algebraic, and verbal representations of functions?
12. Can students find and interpret the regression coefficient in order to determine how well an algebraic model fits given data?
13. Can students explain limitations of mathematical models?

### Unit 1 Grade-Level Expectations (GLEs)

GLE #	GLE Text and Benchmarks
<b>Algebra</b>	
4.	Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)
6.	Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)
7.	Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)
8.	Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)
10.	Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)
<b>Data Analysis, Probability, and Discrete Math</b>	
19.	Correlate/match data sets or graphs and their representations and classify them as exponential, logarithmic, or polynomial functions (D-2-H)
20.	Interpret and explain, with the use of technology, the regression coefficient and the correlation coefficient for a set of data (D-2-H)
22.	Explain the limitations of predictions based on organized sample sets of data (D-7-H)
<b>Patterns, Relations, and Functions</b>	
25.	Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)
27.	Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology (P-3-H)
28.	Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)
29.	Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)

## Sample Activities

### Activity 1: Graphically Speaking (GLEs: 6, 25)

Materials List: Graphically Speaking BLM (one per student), pencils

- Hand out the Graphically Speaking BLM. Students work alone to complete exercise #1.
- Students get in groups of 3-4 to discuss their results and make adjustments.
- Discuss the answers to #1 as a class – see Graphically Speaking with Answers BLM
- Students work together to complete the rest of the Graphically Speaking BLM.
- Students may walk around to compare answers with other groups.
- Discuss any disagreements as a class. Let students defend their answers.
- Step back and let the students try to come to a consensus before presenting the correct answers.
- Assign similar problems for homework.

Extension: Let students create graphs for given function characteristics. For example:

- domain  $[-4, \infty)$  ; range  $(-\infty, \infty)$  ; increasing  $(-1, 0) \cup (4, \infty)$  ; decreasing  $(-\infty, -1)$  ; constant  $(0, 4)$  ;  $f(0) = -3$  ;  $f(-2) = 5$  ;  $f(x) = 0$  when  $x = -2$  and  $x = 1$  ;  $f(x) > 0$  from  $(-1, \infty)$  ;  $f(x) < 0$  from  $(-\infty, -1)$
- Note: graphs will vary

### Activity 2: Family of Functions (GLEs: 6, 25, 27)

Materials List: graphing calculator, Family of Functions BLM (two copies per student), pencils

It is important that students recognize functions based on their respective equations and graphs. As a class, fill in the vertical components of the modified *word grid* ([view literacy strategy descriptions](#)) with the names of various functions. Then, fill in the horizontal components with important function characteristics.

*Word grids* are utilized to elicit student participation in learning important terms and concepts. Therefore, let the students lead the class discussion about function names and properties. If they leave important items out, use leading questions to guide them in the right direction.

- Hand out graphing calculators and the Family of Functions BLMs. Since there are at least ten functions that should be included in this activity, two *word grids* will be needed for each student (five functions per page).
- Ask students to name functions – constant, linear, quadratic, etc. List the function names vertically in the *word grid*. Check the answer key to make sure that all major functions have been identified and included in the *word grid*.
- Ask students to name important function characteristics. List the characteristics horizontally in the *word grid*. Check the Family of Functions with Answers BLM to

make sure that all important characteristics have been identified and included in the *word grid*.

- Let the students work on the rest of the *word grid*, on their own, for about 10 minutes.
- Give the students time to compare answers with each other and make adjustments.
- Discuss the results as a class. Be sure to discuss the similarities and differences between the functions.

Students should keep the Family of Functions *word grid* in their notebooks for future reference. Students should be given time to study the *word grid* before the unit exam. Keep blank *word grids* on hand for students that wish to complete it again on their own as a study guide.

### Activity 3: Translations, Dilations, and Reflections (GLEs: 6, 7, 25, 28)

Materials List: graphing calculator, Translations, Dilations, and Reflections BLM (one per student), pencils

Real-life applications of functions rarely come in parent form (basic function with no operations;  $f(x) = x^3$ ). Therefore, it is essential for students to discover the effects of adding, subtracting, multiplying, and dividing the family of functions by constants. Be sure to discuss what happens to the characteristics of each function (i.e. domain, range, zeros, asymptotes, etc.).

- Allow the students to get in small groups of 3-4.
- Hand out graphing calculators.
- Sketch the absolute value function on the board;  $f(x) = |x|$ .
- Ask the students to graph the absolute value function with the calculator.
- Ask the students to figure out how to move the function to the right 2 units and up 3 units using the calculator.
- Let volunteers record the translation, in function form, on the board:  
 $f(x) = |x - 2| + 3$ .
- Ask the students to graph the cubic function with the calculator;  $f(x) = x^3$ .
- Ask the students to figure out how to move the function to the left 5 units and down 1 unit.
- Let volunteers record the translation, in function form, on the board:  $f(x) = (x + 5)^3 - 1$ .
- If necessary, give more examples. Otherwise, move on to dilations.
- Ask the students to graph the rational function with the calculator;  $f(x) = 1/x$ .
- Ask the students to figure out how to stretch the function vertically by a factor of 2.
- Let volunteers record the dilation in function form on the board:  $f(x) = 2/x$ . NOTE: To help students understand the effects of multiplying the function by 2, write a T-chart for the parent function and the dilation. From the T-charts, students clearly see that the  $x$  values do not change while the  $y$  values are doubled.
- Ask the students to graph the cube root function with the calculator;  $f(x) = \sqrt[3]{x}$ .
- Ask the students to figure out how to shrink the function vertically by a factor of 3.
- Let volunteers record the dilation, in function form, on the board:  $f(x) = 1/3 \sqrt[3]{x}$ . Again, take the time to examine the T-charts of the parent and dilation functions.
- Ask the students to graph the exponential function;  $f(x) = e^x$ .

- Ask the students to figure out how to stretch the exponential function horizontally by a factor of 3.
- Let volunteers record the dilation, in function form, on the board:  $f(x) = e^{\frac{1}{3}x}$ . Again, a T-chart will help the students see that the  $x$  values are tripled while the  $y$  values do not change.
- Let volunteers record the dilation, in function form, on the board:  $f(x) = e^{\frac{1}{3}x}$ . Again, a T-chart will help the students see that the  $x$  values are tripled while the  $y$  values do not change.
- Ask the students to graph the natural log function:  $f(x) = \ln x$ .
- Ask the students to figure out how to shrink the function horizontally by a factor of 2.
- Let volunteers record the dilation, in function form, on the board:  $f(x) = \ln(2x)$ .
- Ask students to graph the quadratic function:  $f(x) = x^2$ .
- Ask the students to figure out how to reflect the function over the  $x$ -axis.
- Let volunteers record the reflection, in function form, on the board:  $f(x) = -x^2$ .
- Ask the students to graph the square root function:  $f(x) = \sqrt{x}$ .
- Ask the students to figure out how to reflect the function over the  $y$ -axis.
- Let volunteers record the reflection, in function form, on the board:  $f(x) = \sqrt{-x}$ .
- Provide additional examples if necessary. Otherwise, assign the Translations, Dilations, and Reflections BLM for homework.

#### Activity 4: In Pieces (GLEs: 4, 8, 10, 25)

Materials List: graphing calculators, In Pieces BLM (one per student), pencils

Piecewise functions are functions that use different rules for different intervals of the domain. The absolute value function can be decomposed into the following piecewise function.

$$f(x) = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases} \quad \text{*NOTE: Zero can either be included in the domain of the first piece or second piece.}$$

There are numerous real-life applications of piecewise functions. The amount of income tax a citizen pays and the cost of mailing a package are just two examples. Students should be able to graph piecewise functions written in function form. Students should also be able to write the function form of piecewise functions from graphs and real-life situations.

- Allow the students to form groups of 3 or 4.
- Introduce piecewise functions using the decomposition of the absolute value function shown above. Be sure to graph the piecewise function and discuss the domain and range. It is helpful to construct a T-chart for each piece of the function.
- Start a discussion about personal income tax. Discuss the differences between flat, progressive, and regressive taxes. Ask students which type of tax seems to be fair.
- Hand out graphing calculators and the In Pieces BLM (one per student).
- Allow each group to work on the In Pieces BLM at its own pace. Walk around the room and assist when necessary.
- It is important to let the groups create their own scales and labels for the independent and dependent variables.

- Ask each group to write one piece of the function on the board. As each student writes a piece of the function, ask that student to identify the piece as linear, quadratic, cubic, etc.
- Ask each remaining group to graph one piece of the function on the board.
- Allow the students to discuss whether or not the tax structure is fair.

Extension: Ask students to create a tax structure model that they think would be better than the one presented in class.

Sample practice and/or homework problems: Have students identify each piece by its function name – linear, quadratic, cubic, square root, constant, and exponential. Remind students to create a T-chart for each piece.

Graph and State the Domain & Range

$$1. f(x) = \begin{cases} 2x - 1 & ; x \leq 0 \\ x^2 + 1 & ; x > 0 \end{cases}$$

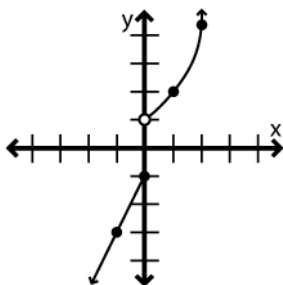
$$2. f(x) = \begin{cases} \sqrt{x} & ; x > 1 \\ -x^3 & ; x < 1 \end{cases}$$

$$3. f(x) = \begin{cases} 3^x & ; x > -1 \\ -1 & ; x \leq -1 \end{cases}$$

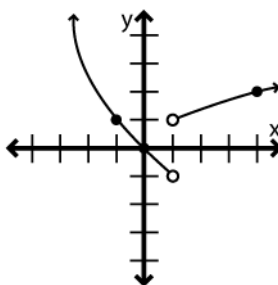
$$4. f(x) = \begin{cases} 4 - x & ; x > 4 \\ 5 & ; 0 \leq x < 4 \\ x - 4 & ; x < 0 \end{cases}$$

**Solutions**

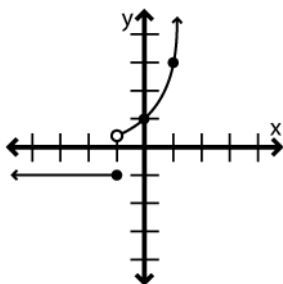
1.



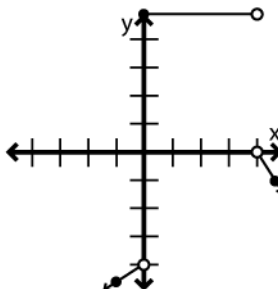
2.



3.



4.



**Activity 5: Function Monsters (GLEs: 4, 6, 28)**

Materials List: graphing calculators, completed Family of Functions BLM, poster boards (one per group), paper, pencils

In this activity, students will enhance their knowledge of combinations of functions. Composite combinations and arithmetic combinations are a part of the algebra II curriculum. Thus, only a brief review should be necessary.

**Sample Review Problems**

$$f(x) = x^2 - 5x - 7 \quad g(x) = 5 - 3x \quad h(x) = x^3 + 11$$

1.  $h(f(g(-2)))$       *Solution: Composite combination ; 205,390*
2.  $f(g(a + 4))$       *Solution: Composite combination ;  $9a^2 + 57a + 77$*
3.  $(h - f)(x)$       *Solution: Arithmetic combination ;  $x^3 - x^2 + 5x + 18$*
4.  $(gh)(5)$       *Solution: Arithmetic combination ; -1360*

- Allow the students to form groups of 2-4.
- Ask each student to get out the completed Family of Functions BLM.
- Each group will select two or more different functions to combine from the Family of Functions BLM. The functions may be in parent, translated, and/or dilated form. For example, one function could be  $f(x) = \sqrt{x}$  (parent form), another function could be  $g(x) = x^2$  (parent form), and a third function could be  $h(x) = -4x+4$  (dilated and translated form).
- Each group will then decide how to combine the functions to create a function monster. For example, one possible combination could be  $m(x) = f(g+h)(x)$ .
- On the poster board, each group must write the chosen functions, the combination function, the domain and range of the combination function, graph the combination function, find the following values if they exist:  $m(0)$ ,  $m(-1)$ , and  $m(10)$ , name the combination function (Harry, Glob, Glenda, etc.), and draw a character for the function monster (Cyclops, alien, robot, etc).
- Hang the posters around the classroom and give the students time to walk around to see the monsters. If time permits, let volunteers present their function monsters to the class.
- See Sample Assessments (at the end of the unit) for a sample grading rubric.

**Activity 6: Zeroing In (GLEs: 6, 8, 27)**

Materials List: graphing calculators

Students should be familiar with solving equations by graphing them and locating the zeros of the function. Briefly review this process by having students solve the following equations by hand. Then, let the students graph the equations by replacing the 0 with  $y$  so that they can

discover that the real zeros are the  $x$ -intercepts of the graph. Make sure the students identify the type of function (linear, quadratic, cubic, exponential, etc.). For polynomial functions, review Descartes' Rule of Signs to determine the number of possible positive and negative real roots before examining the graphs. Remind students that zeros and roots are the real and/or complex values of  $x$  that make  $P(x) = 0$  and that the real zeros are the  $x$ -intercepts.

Example #1      $5x - 4 = 0$      *Solution:  $x = 4/5$  The function is linear. Since the degree of this polynomial function is 1, there is one root. Descartes Rule of Signs shows one sign change since the first term is positive and the second term is negative. This means that there will be one positive root.*

Example #2      $x^2 + 7x + 10 = 0$      *Solution:  $x = -5$  or  $-2$  The function is quadratic. Since the degree of this polynomial function is 2, there are two roots. There are no sign changes since all three terms are positive. This means that both roots are either negative or complex. In this case, the graph showed that they were negative.*

Example #3      $2\sqrt{x} + 1 = 15$      *Solution:  $x = 49$  Since this is not a polynomial function, there is no need to count the number of sign changes. Make sure that the students set the equation equal to zero before graphing it. Let them work together to find a good viewing window.*

These equations were relatively easy to solve algebraically. The following combination of functions is difficult to solve algebraically. Let the students struggle to solve it by hand for a couple of minutes. Then, before too much frustration sets in, ask them if there might be an easier way to solve it. From the first three examples, students should shout out to graph it. Let them graph it and find the solution.

Example #4      $3^x - 2x = 5$      *Solution:  $x = 2$  This is a combination of an exponential and a linear function.*

Let students work in small groups of 2-3 on the following combinations of functions. Tell the students to round all answers to the nearest thousandth when necessary.

- $\frac{x}{1-x} + 4x + \frac{9}{2} = 0$      *Solution:  $x = -1$  This is a combination of a rational, linear, and a constant function.*
- $x^3 \log x - 5x^2 = 500$      *Solution:  $x = 10$  This is a combination of a cubic, logarithmic, quadratic, and a constant function.*
- $5e^x - 3x = 44$      *Solution:  $x \approx -14.667$  or  $x \approx 2.322$  This is a combination of an exponential, linear, and a constant function.*

4.  $(x - 5)^2 + \ln x - 3 = 0$

*Solution:  $x \approx 3.699$  or  $x \approx 6.092$  This is a combination of a quadratic, logarithmic, and a constant function.*

### Activity 7: Inverse Functions (GLEs: 4, 6, 8, 25, 29)

Materials List: graphing calculators, completed Family of Functions BLMs, Inverse Functions BLM (teacher only)

Students have learned to utilize inverse functions to “unwrap” and solve equations. However, many students do not understand the concept of inverse relationships. To create a better understanding of inverse relationships, this activity will utilize *split-page notetaking* ([view literacy strategy descriptions](#)). Organization in mathematics does matter. Therefore, demonstrate *split-page notetaking* on the board by drawing a vertical line splitting the page into one-third on the left and two-thirds on the right. On the left-hand side, have the students write the main heading Verbal Representations. The examples and any other additional notes should be written on the right-hand side of the page. Depending on the size of the handwriting, students will have to repeat the process for 2-5 pages since there are four main headings. See the Inverse Functions BLM for an example of *split-page notetaking*. At the end of the activity, demonstrate for students how they can study from these notes by covering one column, and then using the information in the other column to recall the covered information.

#### Verbal Representations

Example #1: amount you pay for gas                      number of gallons purchased

The total cost of the gas is dependent on the number of gallons purchased. The ordered pairs for this function are (number of gallons, total cost). Now, reverse the relationship. The number of gallons of gas you can purchase depends on the total amount of money you have available for gas. The ordered pairs for this inverse function are (total cost, number of gallons). Note that the domain and range of the original relationship become the range and domain of the inverse relationship.

Let the students write the original relationship and inverse relationship for the next example.

Example #2: number of hours worked                      amount of paycheck

If you are paid by the hour, the total amount of your paycheck is dependent on the number of hours you worked that pay period. The ordered pairs for this function are (number of hours worked, amount of paycheck). Now, reverse the relationship. The number of hours you work is dependent upon the money you need to earn. The ordered pairs for this inverse function are (amount of paycheck, number of hours worked). Again, make sure that the students understand that the domain and range of the original function become the range and domain of the inverse function.

Now, examine the following relationship that does not result in two functions.

Example #3: height of a punted football      number of seconds since it was kicked

The height of the punted football is dependent on the number of seconds that have elapsed since the ball was kicked. The ordered pairs for this function are (time, height). Since everything that goes up must come down, the inverse relationship will not be a function. There will be two different times when the ball is at a particular height. Sketch the parabola to illustrate this concept. This is a good time to introduce the horizontal line test and one-to-one functions. Ask the students to state another relationship that will not have an inverse that is a function. Have them justify their answers using the horizontal line test.

### Graphic Representations

Let the students work with partners to determine which of the functions from the Family of Functions BLM have inverses that are also functions. Discuss the results as a class and make the students justify their selections. Be sure that their answers include the following functions: linear, cubic, square root, cube root, rational ( $f(x) = 1/x$ ), exponential, and logarithmic. Ask the students if they notice any characteristics that each of these functions share. Hopefully, someone will notice that these functions either always increase or decrease throughout their domains. Remind students that inverse functions are reflections about the line  $y = x$ , since the domain becomes the range and the range becomes the domain. Let students come up to the board to sketch the original function, the line  $y = x$ , and the inverse function.

### Tabular Representations

Ask the students to determine which of the functions below have inverses that are also functions. If there is an inverse, ask the students to create a table for it.

1. 

$x$	-1	0	1	3	5
$f(x)$	3	2	5	1	-3

*Solution: The inverse is not a function. The function decreases, increases, and decreases again. Thus, it cannot be a one-to-one function.*

2. 

$x$	-3	-2	-1	0	1
$f(x)$	-1	0	3	2	1

*Solution: The inverse is a function since the function is increasing throughout the domain.*

$x$	-1	0	3	2	1
$f^{-1}(x)$	-3	-2	-1	0	1

3. 

$x$	-2	0	2	4	6
$f(x)$	6	3	0	-3	-6

*Solution: The inverse is a function since the function is decreasing throughout its domain.*

$x$	-6	-3	0	3	6
$f^{-1}(x)$	6	4	2	0	-2

### Algebraic Representations

To find the inverse function from an equation, swap the  $x$  and the  $y$ , and then solve for  $y$ .

Ask the students to work in small groups to find the inverse function, if it exists, for each problem below. Students should justify their answers using composite functions:  $f(g(x)) = x$  and  $g(f(x)) = x$ .

1.  $f(x) = 2x + 5$                       *Solution:*  $f^{-1}(x) = \frac{x-5}{2}$

2.  $f(x) = x^2 - 4$                       *Solution:* *The inverse is not a function since the parabola does not pass the horizontal line test.*

3.  $f(x) = \sqrt[3]{x-1}$                       *Solution:*  $f^{-1}(x) = x^3 + 1$

4.  $f(x) = \frac{1}{x+3}$                       *Solution:*  $f^{-1}(x) = \frac{1-3x}{x}$

### Activity 8: Modeling with Functions (GLEs: 19, 20, 22)

Materials List: graphing calculators

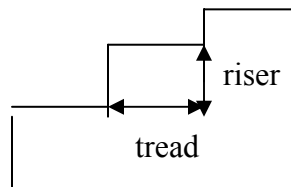
Modeling with functions will be a recurring theme throughout the school year. In this activity students will learn how to model real-life situations using functions in tabular, graphic, algebraic, and verbal forms.

#### Example #1

Stairs should be constructed according to safety guidelines. The normal tread/pace length is about 60 cm. This distance must be decreased by 2 cm for every 1 cm that a person’s foot is raised when climbing stairs. According to these guidelines, what is the relationship between the tread length and the riser?

Help the students construct a tabular representation, but let them fill it in.

Riser (cm)	0	1	2	3
Tread (cm)	60	58	56	54



Students should recognize the riser length as the independent variable and the tread length as the dependent variable. Students should also notice that the rate of change is -2. Since the rate of change is a constant, students should determine that this is a linear situation.

Next, students should enter the data into the graphing calculator in order to examine the scatter plot. If the students have not already figured out the algebraic model, guide the students through the process of finding the line of best fit. This will generate the model  $y = -2x + 60$  or  $T = -2R + 60$ . Discuss what it means to have a correlation coefficient of -1 (perfect negative linear relationship). Discuss the domain and range of the model. Since both the domain and the range can have lengths between the data values, like 2.5 cm and 3.8 cm, students need to identify the model as being continuous in nature. Ask the students to make predictions about the tread length given various riser lengths and vice versa. Don't forget to state limitations for the mathematical model. For example, the riser values can only be so large; otherwise, no one would be able to climb the stairs.

### Example #2

Amanda collected the following data after the launch of her model rocket.

Time (sec)	0	1	2	3	4	5
Height (ft)	5	85	135	155	140	95

Since the tabular representation is given, students should move on to the graphic representation by entering the data in the calculator and drawing the scatter plot. After examining the scatter plot, ask the students which type of function would best model the data. The parabolic shape will lead the students to the quadratic function. The algebraic model should be  $h \approx -15.8t^2 + 97.2t + 4.5$ . Discuss the domain and range as well as the continuous nature of the model. Discuss the meaning of the correlation coefficient (not a perfect fit but pretty close). Use the model to make predictions about the height given various times after the launch and vice versa.

Extension: Let the students collect height and shoe size data from 6 different classmates. Students should present their data in tabular form, examine the scatter plot, and find the algebraic model. Algebraic models will vary based on the data collected. It is very important to discuss the domain and range of the model because the independent variable (shoe size) is discrete in nature. Shoe sizes only come in whole number and/or half sizes. This makes the model discrete in nature (i.e. the points should not be connected since the domain has specific restrictions).

## Sample Assessments

### General Assessments

- Each student will create a *learning log* ([view literacy strategy descriptions](#)) to use throughout the school year. A *learning log* is a journal students use to record their thoughts, ideas, and reflections about a topic. The first entry will have the title, “Everything I Know About Functions.” This assessment piece can be formal or informal in nature. Informally, students would be assigned a grade for writing the entry. This will let you know what students do and do not understand about functions. Formally, students can be graded based on the inclusion of main topics like definition, notation, domain, range, independent variable, dependent variable, continuous, discrete, as well as finding zeros, inverses, and models from data.
- Each student will create a portfolio throughout the school year. A portfolio is a collection of student work used to demonstrate student understanding of major topics studied. One entry for the portfolio will be a glossary of important vocabulary terms. Students should begin constructing their glossaries on computers so that they can be arranged in alphabetical order as the year progresses.
- Each student will complete a “spiral” comprised of between 5 and 10 problems. A “spiral” is a cumulative collection of problems that students have already studied. Students work the problems on their own. If they need help, students may ask the teacher or another student for assistance. Collaboration is encouraged. Copying is not allowed. Since this is the first unit of the year, “spiral” problems should center on the topics learned in this unit and at least half should be in multiple choice form. This will help prepare students for the unit test and the ACT/SAT exams. “Spirals” should be assigned every two or three weeks.

### Activity-Specific Assessments

- Activity 4: In Pieces

The NASA Pathfinder is an unmanned aircraft. It is controlled manually from a station on the ground. The Pathfinder runs on solar power, but has a battery it can use if necessary. A 1997 Pathfinder mission collected the following data on the battery’s state of charge.

Flight Time (hrs)	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Battery Charge (%)	100	100	100	100	100	100	100	100	100	100	100	92.5	85	77.5

Let the students work in small groups of 2-3. Give the students 5 points for being able to graph this piecewise function. Give the students 10 points for being able to write the piecewise function

correctly – 5 points for the first piece and its domain & 5 points for the second piece and its domain. Go to the following web site for the solutions.

<http://www.ed.arizona.edu/ward/Pathfinder/Battery/batt.html>

➤ Activity 5: Function Monsters

Below is a sample of a rubric that could be used to grade the Function Monsters.

	<b>0 points</b>	<b>1 point</b>	<b>2 points</b>
<b>Two or more original functions</b>	No functions given	Only 1 function given	Two or more functions given
<b>Combination function</b>	No combination given	Combination given but it is incorrect	Correct combination given
<b>Domain of combination function</b>	No domain given	Domain given but it is incorrect	Correct domain given
<b>Range of combination function</b>	No range given	Range given but it is incorrect	Correct range given
<b>Graph of combination function</b>	No graph given	Graph given but it is incorrect	Correct graph given
<b><math>m(0)</math></b>	No value given	Value given but it is incorrect	Correct value given
<b><math>m(-1)</math></b>	No value given	Value given but it is incorrect	Correct value given
<b><math>m(10)</math></b>	No value given	Value given but it is incorrect	Correct value given
<b>Name of function monster</b>	No name given	-----	Name given
<b>Drawing of function monster</b>	No name given	-----	Name given

➤ Activity 8: Modeling with Functions

Groups of 3 or 4 students will collect at least 6 data values for two variables of their choosing (examples: shoe size and height, age and memory recall of items, month and rainfall, etc.). Each group will turn in a paper/poster including the topics listed in the sample rubric below.

	<b>0 points</b>	<b>1-4 points</b>	<b>5 points</b>
<b>Two variables</b>	Do not form a function	-----	Form a function
<b>Tabular representation of data</b>	No data given	Fewer than six data values for each variable	Six data values given for each variable
<b>Graphical model</b>	No labels	Labels given but are incorrect	Correct labels given
<b>Graphical model</b>	No model drawn	Model drawn but not correctly	Correct model drawn
<b>Graph of combination function</b>	No graph given	Graph given but it is incorrect	Correct graph given
<b>Algebraic model</b>	No model given	Model given but it is incorrect	Correct model given
<b>Domain</b>	No domain given	Domain given but it is incorrect	Correct domain given
<b>Range</b>	No range given	Range given but it is incorrect	Correct range given
<b>Predictions given the independent variable</b>	No predictions given	Predictions given but are incorrect	Correct predictions given
<b>Limitations of the model</b>	No limitations given	Limitations are presented but are incorrect or missing main considerations	Correct limitations given and explained properly

**Advanced Math – Functions and Statistics**  
**Unit 2: Triangle Trigonometry**



**Time Frame:** Approximately 2.5 weeks

**Unit Description**

This unit covers all aspects of triangle trigonometry. It begins with a review of right triangle ratios. The Laws of Sines and Cosines are introduced so that problems involving oblique triangles can be solved. The unit ends with real-life applications of triangle trigonometry.

**Student Understandings**

Students will know how to solve triangles using either right triangle ratios or the Laws of Sines and Cosines. They will be able to write angle measurements in decimal form and in degrees, minutes, and seconds. Students will be able to model and solve real-life problems using triangles and trigonometry.

**Guiding Questions**

1. Can students express angles in decimal form as well as in degrees, minutes, and seconds?
2. Can students use right triangle trigonometry to solve right triangles?
3. Can students model and solve real-life problems involving right triangles?
4. Do students recognize when given information results in a unique triangle and when it does not?
5. Can students use the Law of Sines and the Law of Cosines to solve oblique triangles?
6. Can students use the Law of Sines and the Law of Cosines to model and solve real-life application problems?
7. Can students model and solve real-life situations involving vectors?

**Unit 2 Grade-Level Expectations (GLEs)**

GLE #	GLE Text and Benchmarks
<b>Geometry</b>	
<b>Grade 10</b>	
11.	Determine angle measurements using the properties of parallel, perpendicular, and intersecting lines in a plane (G-2-H)
12.	Apply the Pythagorean Theorem in both abstract and real-life settings (G-2-H)
18.	Determine angle measures and side lengths of right and similar triangles using Trigonometric ratios and properties of similarity, including congruence (G-5-H) (M-4-H)

	<b>Grade 11</b>
11.	Calculate angle measures in degrees, minutes, and seconds (M-1-H)
14.	Use the Law of Sines and the Law of Cosines to solve problems involving triangle measurements (M-4-H)

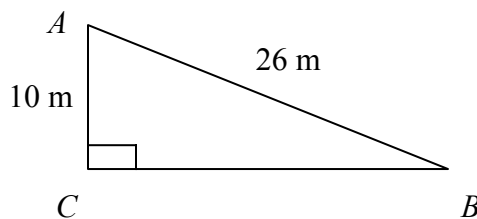
### Sample Activities

#### Activity 1: Solving Right Triangles (GLEs: Grade 10: 12, 18 ; Grades 11-12: 11)

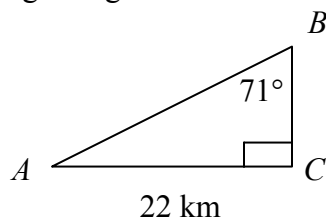
Materials List: calculators, paper, pencil, Solving Right Triangles BLM

This activity builds on students' knowledge of trigonometric ratios learned in geometry.

- Ask students how many measurements can be taken from one triangle.  
*6 measurements: 3 sides and 3 angles*
- Tell students that identifying all 6 measurements is called solving a triangle.
- Ask students how many measurements must be known in order to determine the others.
- Let volunteers come to the board to share their answers.  
*3 measures need to be given and at least one of the measures must be a side*
- Draw the following triangle on the board.

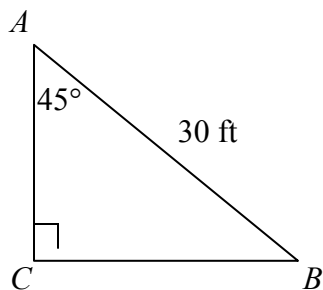


- Ask students which measurements are known.  $C = 90^\circ$ ;  $b = 10\text{ m}$ ;  $c = 26\text{ m}$
- Let students work in small groups to find the other three measurements.
- Let volunteers share their methods and results with the class.  
*To find  $a$ , students could use the Pythagorean Theorem or note that the sides are multiples of the 5-12-13 Pythagorean triple. Since all three sides are now known, students could use the sine, cosine, or tangent ratios to find angle  $A$ .  $A \approx 67.4^\circ$  or  $A \approx 67^\circ 22' 49''$ . Trig ratios are not needed to find  $m\angle B$ , just subtract  $\angle A$  from  $90^\circ$  since the acute angles of a right triangle are complementary. Remind students that  $60'' = 1'$  and  $60' = 1^\circ$ . Thus,  $B \approx 89^\circ 59' 60'' - 67^\circ 22' 49'' \approx 22^\circ 37' 11''$*
- Draw the following triangle on the board.



- Ask students which measurements are known.  $B = 71^\circ$ ;  $C = 90^\circ$ ;  $b = 22 \text{ km}$
- Let students work in small groups to find the other three measurements.
- Let volunteers share their results and methods with the class.
 

$A = 19^\circ$ ;  $a \approx 12.2 \text{ km}$ ;  $c \approx 23.3 \text{ km}$
- Next, move on to the  $45^\circ$ - $45^\circ$ - $90^\circ$  triangle.
- Let students discover why the square root of 2 is always associated with the  $45^\circ$ - $45^\circ$ - $90^\circ$  triangle by drawing three  $45^\circ$ - $45^\circ$ - $90^\circ$  triangles of different sizes on the board. Let students come to the board and create measurements for the legs.
- Ask different groups of students to find the hypotenuse for one of the three triangles.
- Let volunteers share their work and results with the class.
- Ask students if they see a pattern.  $\text{leg} = \text{leg}$ ;  $\text{hyp} = \text{leg}\sqrt{2}$
- Draw three more  $45^\circ$ - $45^\circ$ - $90^\circ$  triangles of different sizes on the board, but this time label only the hypotenuses.
- Ask different groups of students to find the legs for one of the three triangles using the Pythagorean Theorem.
- Ask students if they see a pattern.  $\text{leg} = \text{hyp}/\sqrt{2}$  or  $\text{leg} = \frac{1}{2} \text{hyp} \cdot \sqrt{2}$
- Draw the following triangle on the board, and ask small groups to solve it.



- Let volunteers share their methods and results with the class.
 

$B = 45^\circ$ ;  $a = 15\sqrt{2} \text{ ft}$ ;  $b = 15\sqrt{2} \text{ ft}$
- Now, move on to the  $30^\circ$ - $60^\circ$ - $90^\circ$  triangle.
- Let students discover the patterns of the legs by drawing three  $30^\circ$ - $60^\circ$ - $90^\circ$  triangles of different sizes on the board. Let students come to the board and create measurements for the short legs.
- Ask different groups of students to use the trig ratios to find the long leg and the hypotenuse for one of the three triangles.
- Let volunteers share their work and results with the class.
- Ask students if they see a pattern.  $\text{long leg} = \text{short leg} \cdot \sqrt{3}$ ;  $\text{hyp} = \text{short leg} \cdot 2$
- Draw three more  $30^\circ$ - $60^\circ$ - $90^\circ$  triangles of different sizes on the board. For one triangle, label one acute angle and the short leg. For another triangle, label one acute angle and the hypotenuse. For the third triangle, label one acute angle and the long leg.
- Ask volunteers to share their results with the class.
- Hand out the Solving Right Triangles BLM.
- Allow students to work together to solve each triangle.

- If, at the end of the activity, students are comfortable with solving right triangles, let them turn in the Solving Right Triangles BLM for a grade. See Sample Assessments for scoring guidelines.

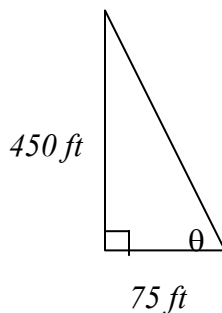
### Activity 2: Applications Involving Right Triangles (GLEs: Grade 10: 18 ; Grades 11-12: 11)

Materials List: calculators, Applications of Right Triangles BLM, paper, pencils

A common use for trigonometry is to measure angles and distances that are either awkward or impossible to measure by ordinary means. Finding heights of buildings, trees, and mountains, distances between objects, vector components, as well as angles of elevation and depression are a few examples.

- Write each of the following problems on the board. Ask students to draw a model, label the known information, write the trigonometric ratio needed, and solve the problem using a calculator.
- Allow students to share methods and compare results.
- Ask volunteers to come to the board and share their methods and results with the class.

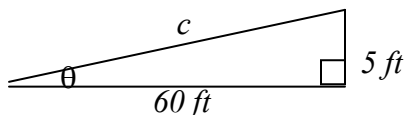
- At 2 p.m. a building 450 feet high casts a shadow 75 feet long. What is the angle of elevation of the sun?



$$\begin{aligned}\tan \theta &= 450/75 \\ \theta &= \tan^{-1}(450/75) \\ \theta &\approx 80.5^\circ\end{aligned}$$

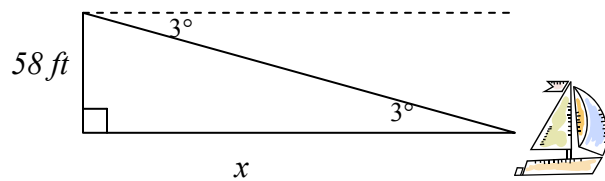
*\*Note: students should know that  $\theta > 45^\circ$  since  $450 > 75$*

- The city plans to install a wheelchair ramp at the entrance to its courthouse. The ramp should have a running slope of 1:12 units. The final height of the ramp is 5 feet. What angle of elevation will the ramp have and how long will the ramp be?



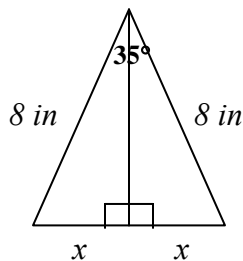
$$\begin{aligned}\tan \theta &= 5/60 & c^2 &= 5^2 + 60^2 \\ \theta &= \tan^{-1}(5/60) & c^2 &= 3625 \\ \theta &\approx 4.8^\circ \approx 4^\circ 45' 49'' & c &\approx 60.2 \text{ ft}\end{aligned}$$

3. A lighthouse keeper observes that there is a  $3^\circ$  angle of depression between the horizontal line of sight and the diagonal line of sight to a ship. If the keeper is 58 feet above the water, how far is the ship from shore?



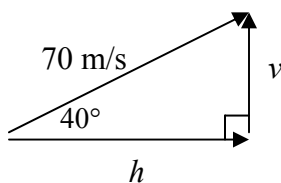
$$\begin{aligned} \tan 87^\circ &= x/58 \\ x &= 58 \tan 87^\circ \\ x &\approx 1,106.7 \text{ ft} \end{aligned}$$

4. The legs of an isosceles triangle are 8 inches long and the angle between them is  $35^\circ$ . What is the length of the third side?



$$\begin{aligned} \sin 17.5^\circ &= x/8 \\ x &= 8 \sin 17.5^\circ \\ x &\approx 2.4 \text{ in.} \\ 2x &\approx 4.8 \text{ in.} \end{aligned}$$

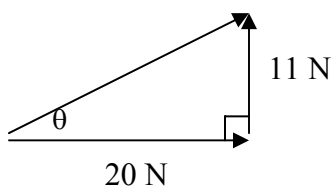
5. A ball is thrown with an initial velocity of 70 m/s at an angle of  $40^\circ$  with the horizontal. Find the vertical and horizontal components of the velocity vector.



$$\begin{aligned} \cos 40^\circ &= h/70 & \sin 40^\circ &= v/70 \\ h &= 70 \cos 40^\circ & v &= 70 \sin 40^\circ \\ h &\approx 53.6 \text{ m/s} & v &\approx 45.0 \text{ m/s} \end{aligned}$$

\*Note: make sure students know that a vector is a quantity possessing both magnitude and direction. Students will also study vectors in physics.

6. Find the direction and magnitude of a force whose horizontal and vertical components are 20 Newtons and 11 Newtons.



$$\begin{aligned} \text{mag}^2 &= 11^2 + 20^2 & \tan \theta &= 11/20 \\ \text{mag}^2 &= 521 & \theta &= \tan^{-1}(11/20) \\ \text{mag} &\approx 22.8 \text{ N} & \theta &\approx 28.8^\circ \approx 28^\circ 48' 39'' \end{aligned}$$

Note: students should know that  $\theta < 45^\circ$  since  $11 < 20$ .

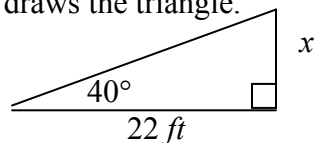
- To end the activity, hand out the Applications of Right Triangles BLM.
- Allow students to form groups of 3.
- Students will demonstrate mastery of applications involving right triangles by employing the use of *story chains* ([view literacy strategy descriptions](#)). The first group member will

write an application problem involving finding the height of an object. The second group member will draw and label the triangle. The third group member will solve the problem. The first group member will check the results and provide feedback.

- Provide the following example of a story chain to help get the students started.

The first student writes: Find the height of a flag pole that casts a shadow of 22 ft with an angle of elevation of  $40^\circ$ .

The second group member draws the triangle.



The third group member solves the problem.

$$\tan 40^\circ = x/22$$

$$x = 22 \tan 40^\circ$$

$$x \approx 18.5 \text{ ft}$$

- If all three group members agree with the model and results of the first problem, the second group member will write an application problem involving finding either an angle of elevation or an angle of depression. The other two group members will solve the problem. The second group member will check the result and provide feedback.
- If all three group members agree with the model and results of the second problem, the third group member will write an application problem involving finding either the horizontal and vertical components of a vector or the magnitude and direction of a vector. The other two group members will solve the problem. The third group member will check the result and provide feedback.
- Each group could turn in the BLM for a 15-point grade (5 points per problem). See Sample Assessments for scoring guidelines.

### Activity 3: Law of Sines (GLEs: 11, 14)

Materials List: Discovering the Law of Sines BLM, Law of Sines: Split-Page Notetaking BLM (teacher only), paper, pencils

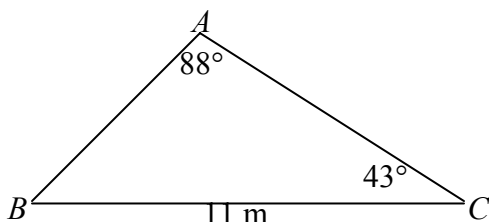
In this activity, students will discover the Law of Sines. They will use altitudes to create right triangles from oblique triangles. Then, students will use their knowledge of right triangle trigonometric ratios to generate the Law of Sines.

After discovering the Law of Sines, students will use it to solve oblique triangles. Since there are so many possible conditions, *split-page notetaking* ([view literacy strategy descriptions](#)) will be utilized to help students organize and understand the material presented

- Hand out the Discovering the Law of Sines BLM.
- Work problems 1 – 6 as a class.
- Let small groups complete the rest of the BLM together.
- Discuss the results as a class.
- Ask the class to name all of the possibilities for the three triangle measurements needed to solve a triangle. *SAS, SSS, SSA, AAS, ASA*
- Ask the class which of the possibilities can be solved using the Law of Sines.

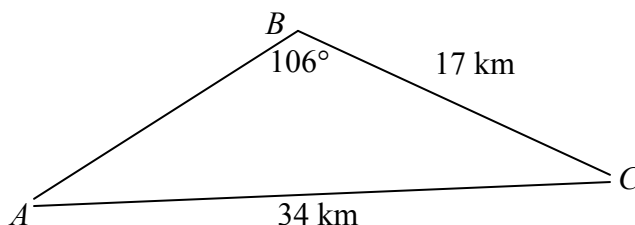
*For proportions, at least three of the quantities must be known. Therefore, the Law of Sines can be used when the following measures are known: SSA, AAS, & ASA.*

- Ask the class under which of the above condition(s) is a unique triangle formed.  
*If two angles are known, there is only one possibility for the third angle. Therefore, measurements of the forms AAS and ASA will result in unique triangles.*
- Draw the following oblique triangle on the board for students to solve.



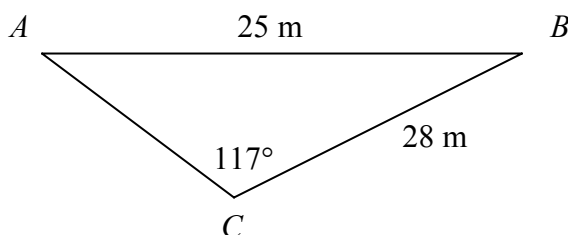
$$B = 49^\circ ; b \approx 8.3\text{ m} ; c \approx 7.5\text{ m}$$

- Solve the triangle as a class utilizing *split-page notetaking*. Demonstrate *split-page notetaking* on the board by drawing a vertical line splitting the page into one-third on the left and two-thirds on the right. On the left-hand side, have the students write the heading Law of Sines – AAS, unique triangle, and draw the triangle. The proportions and results should be written on the right-hand side of the page. Refer to the Law of Sines Split-Page Notetaking BLM to get the students started.
- Let volunteers share their methods and results with the class.
- Draw the following triangle on the board for students to solve.



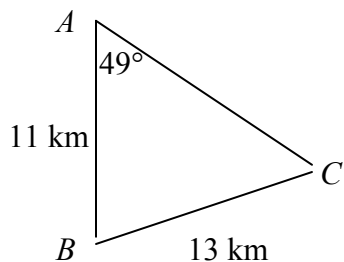
$$\begin{aligned} A &\approx 28.7^\circ \approx 28^\circ 43' 36'' \\ C &\approx 45.3^\circ \approx 45^\circ 16' 24'' \\ c &\approx 25.1\text{ km} \end{aligned}$$

- Have the students write the heading *SSA – Obtuse Angle* and draw the triangle on the left-hand side of the paper. The proportions and the results should be written on the right-hand side.
- Ask students if the measurements will result in a unique triangle.  
*Since there can only be one obtuse angle in a triangle, the measurements will result in a unique triangle.*
- Let volunteers share their methods and results with the class.
- Draw the following triangle on the board for students to solve.



*No triangle: the largest angle does not have the largest side.*

- Have the students write the heading *SSA – Obtuse Angle* and draw the triangle on the left-hand side of the paper. The proportions and the results should be written on the right-hand side.
- Draw the following triangle on the board for students to solve.



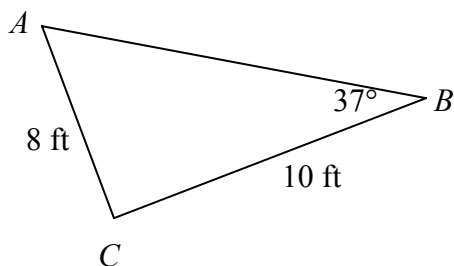
$$C \approx 39.7^\circ \approx 39^\circ 41' 15''$$

$$B \approx 91.3^\circ \approx 91^\circ 18' 45''$$

$$b \approx 17.2 \text{ km}$$

No second triangle: the sum of  $C$ 's supplement and angle  $A$  is greater than  $180^\circ$ .

- Ask the class if the measurements will result in a unique triangle.  
*Maybe: Since only one acute angle is given, the  $\sin A$  generates two possibilities for angle  $C$ . One possibility is an acute angle. The other possibility is the obtuse supplement because  $\sin C = \sin(180^\circ - C)$ . If the sum of the supplement and angle  $A$  does not equal or exceed  $180^\circ$ , then two triangles will exist. If the sum  $\geq 180^\circ$ , only one triangle will exist.*
- Have the students write the heading *SSA – Acute Angle* and draw the triangle on the left-hand side of the paper. The proportions and the results should be written on the right-hand side.
- Let volunteers share their results and methods with the class.
- Draw the following triangle on the board for students to solve.



First Triangle:

$$A \approx 48.8^\circ \approx 48^\circ 47' 14''$$

$$C \approx 94.2^\circ \approx 94^\circ 12' 46''$$

$$c \approx 13.3 \text{ ft}$$

Second Triangle:

$$A \approx 131.2^\circ \approx 131^\circ 12' 46''$$

$$C \approx 11.8^\circ \approx 11^\circ 47' 14''$$

$$c \approx 2.7 \text{ ft}$$

- Ask the class if the measurements will result in a unique triangle.  
*No: Again, only one acute angle is given. The  $\sin A$  generates two possibilities for angle  $A$ . One possibility is an acute angle. The other possibility is the obtuse supplement because  $\sin A = \sin(180^\circ - A)$ .*
- Have the students write the heading *SSA – Acute Angle* and draw the triangle on the left-hand side of the paper. The proportions and the results should be written on the right-hand side.
- Let volunteers share their methods and results with the class.
- Allow students to form groups of 4-6.
- Assign each group one of the following triangles to create and solve.
  - 1) *AAS*
  - 2) *SSA – Obtuse Angle*
  - 3) *SSA – Obtuse Angle – Measurements result in no triangle*
  - 4) *SSA – Acute Angle – Measurements result in one triangle*
  - 5) *SSA – Acute Angle – Measurements result in two triangles.*
- Ask one member from each group to draw his or her group's triangle on the board. Do not let students write answers on the board yet.

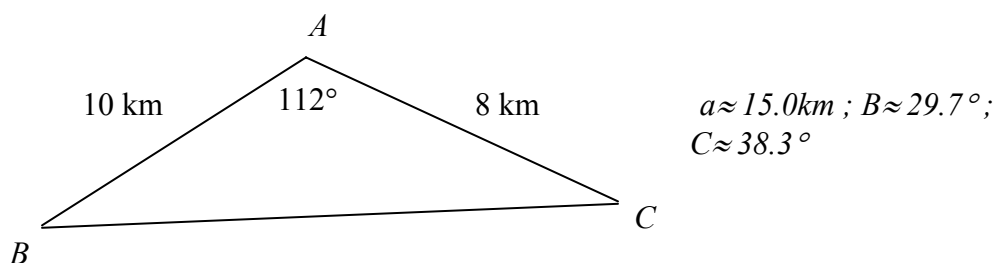
- Each student will solve the triangles for homework.
- The next day, one member from each group will write the answers to his or her group's respective triangle on the board, so that each student can check his or her own homework.
- At the end of the activity, demonstrate for students how they can study from split-page notes by covering one column, and then using the information in the other column to recall the covered information.

#### Activity 4: Law of Cosines (GLE: 14)

Materials List: calculators, Discovering the Law of Cosines BLM, paper, pencils

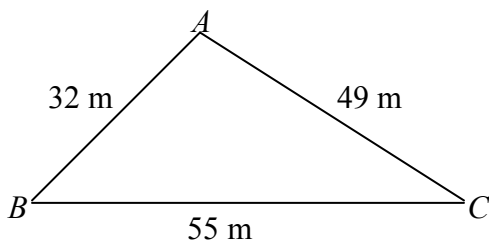
In this activity, students will continue to solve oblique triangles as a warm-up for the applications to follow in Activity 5.

- Ask students which groups of three triangle measurements cannot be solved using the Law of Sines. *SAS & SSS*
- Allow students to form groups of 3-4.
- Hand out the Discovering the Law of Cosines BLM.
- Get students started by working #1 as a class. Then, walk around and assist groups when necessary.
- When all groups have finished, discuss the results as a class. Ask students if it is necessary to remember all six equations.  
*No: If one equation is known, patterns and algebra can be used to find the other equations.*
- Draw the following triangle on the board for students to solve.



- Employing the use of *split-page notetaking* ([view literacy strategy descriptions](#)), have students write the heading, Law of Cosines. In the left column, students will write *SAS* and draw the triangle.
- Students will write the appropriate equations and results in the right column.
- Note: Students will ask if they can use the Law of Sines once side *a* has been found. The Law of Sines can be used to solve the rest of the triangle, especially since an obtuse angle was given. The Law of Sines can also be used after finding the corresponding side of a given acute angle, but students have to remember that two possibilities will exist for the second angle since  $\sin \theta = \sin (180^\circ - \theta)$ . To avoid this problem, students can use the Law of Cosines to find the second angle.

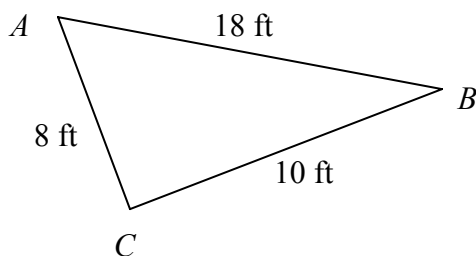
- Let volunteers share their methods and results with the class.
- Draw the following triangle on the board for students to solve.



$$A \approx 82.7^\circ ; B \approx 62.1^\circ ; C \approx 35.2^\circ$$

Note: Only continue to convert from *DD* to *DMS* if students need additional practice.

- Have students label the left column as *SSS* and draw the triangle.
- Students will write the equations and the results in the right column.
- Let volunteers share their methods and results with the class.
- Draw the following triangle on the board for students to solve.



*No triangle: the triangle inequality theorem states that the sum of any two sides must be greater than the third side.*

- Have students label the left column as *SSS* and draw the triangle.
- Students will write the equations and the results in the right column.
- Let volunteers share their methods and results with the class.
- To end the activity, let each student create and solve two triangles. One triangle should have the measurements *SAS*, and the other triangle should have the measurements *SSS*. Note: Students must create triangles that exist and their solutions should be written on a separate sheet of paper.
- Let students swap triangles. Each student will solve the triangles for homework.
- The next day, let students swap papers back with the original owner so that he or she can check the answers.

### Activity 5: Applications of Oblique Triangles (GLEs: Grade 10: 11 ; Grades 11-12: 14)

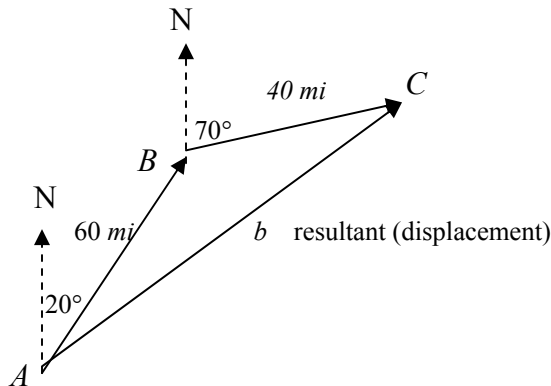
Materials List: calculators, Applications of Oblique Triangles BLM, paper, pencils

In this activity, students will use their knowledge of the Law of Sines and the Law of Cosines to solve real-life application problems. To assist students in making connections between math and physics, some application problems will involve adding vectors using the tip-to-tail method. The activity will begin with resultant vector situations. Students should be able to handle the rest of the Applications of Oblique Triangles BLM on their own.

- Write the following problem on the board.

A ship sails 60 miles on a bearing of  $20^\circ$  and then turns and sails 40 miles on a bearing of  $70^\circ$ . Find the magnitude and direction of the displacement (resultant) vector.

- Ask students what bearing means. *Angle measured clockwise from North.*
- Ask students to model the path and displacement of the ship using vectors.
- Let volunteers share their results with the class.

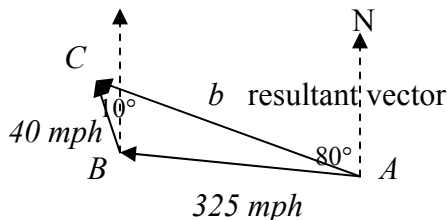


- Ask students to label the vertices of the triangle as shown above. This will make discussions about the problem easier to follow.
- Ask students which components of the model are parallel. *the North rays*
- Ask students to describe what would happen to the  $70^\circ$  angle if the first vector was extended. *A corresponding angle of  $20^\circ$  would be formed. The other angle would have to be  $70^\circ - 20^\circ$ , which is  $50^\circ$ .*
- Draw and label the angles formed by the extension.
- Ask students to find the  $m\angle ABC$ . *Linear pair:  $180^\circ - 50^\circ = 130^\circ$*
- Ask students which Law (Sines or Cosines) should be used to find  $b$ . *Since this is a SAS triangle, the Law of Cosines should be used.*
- Ask students to find the magnitude (length) of the resultant vector.
- Let volunteers share their results with the class.  $b \approx 91.0$  miles
- Ask students to find the direction (bearing) of the resultant vector.
- Note: Now that side  $b$  is known, students can use either the Law of Sines or the Law of Cosines to find  $m\angle BAC$ .
- Let volunteers share their methods and results with the class.  $m\angle BAC \approx 19.7^\circ$ ; Thus, the bearing is  $20^\circ + 19.7^\circ \approx 39.7^\circ$ .
- Write the following problem on the board.

A plane is flying  $80^\circ$  west of North at 325 mph. A tailwind is blowing  $10^\circ$  west of North at 40 mph. Find the actual velocity and direction of the plane.

- Ask students to model the path of the plane using vectors.
- Let volunteers share their results with the class.

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- Ask students to label the vertices of the triangle as shown above. This will make discussions about the problem easier to follow.
- Again, ask students which components of the model are parallel. *the North rays*
- Ask students to describe what would happen if the first vector was extended.
  - A corresponding angle of  $80^\circ$  would be formed, but it would be divided into a  $10^\circ$  angle and a  $70^\circ$  angle.*
- Draw and label the angles formed by the extension.
- Ask students to find the  $m\angle ABC$ . *Linear pair:  $180^\circ - 70^\circ = 110^\circ$*
- Before students find the velocity of the plane, ask them if the answer should be greater than or less than 325 mph.
  - Greater than 325 mph since the wind is a tailwind.*
- Let volunteers share their methods and results with the class.
  - $b \approx 340.8$  mph*
- Ask students to find the direction of the plane.
- Note: Now that side  $b$  is known, students can use either the Law of Sines or the Law of Cosines to find  $m\angle BAC$ .
- Let volunteers share their results with the class.
  - $m\angle BAC \approx 6.3^\circ$  ; Thus, the plane is traveling  $80^\circ - 6.3^\circ \approx 73.7^\circ$  west of North.*
- If additional examples are needed, let the students create the problems and then guide them through both the modeling process and the solving process.
- Hand out the Applications of Oblique Triangles BLM.
- Allow students to complete the Applications of Oblique Triangles BLM in groups of 3-4.
- Walk around and assist if necessary.
- At the end of the activity, let each group select 4 of the 6 problems to turn in for a grade. Selected problems should be marked with asterisks.
- If time permits, rework the vector problems using the parallelogram method by placing the vectors tail-to-tail.

## Sample Assessments

### General Assessments

- Each student will create a sixth entry for his or her *learning log* ([view literacy strategy descriptions](#)). Remember, students will be adding entries to the *learning logs* throughout the school year. The sixth entry will have the title, “Triangle Trigonometry in the Real World.” Requirements for this entry will include at least three careers that use triangle trigonometry and a worked example from each career. This assessment piece can be formal or informal in nature. Informally, students would be assigned a grade for writing

the entry. This will let you know what students do and do not understand about triangle trigonometry and its real-life applications. Formally, students can be graded based on the inclusion of at least three valid career choices and correctly worked examples for each one.

- Each student will continue to add terms from this unit to the glossary started in unit one. This glossary will be included in a student portfolio that will be graded at the end of each semester.
- Each student will complete a “spiral” comprised of 5 -10 teacher-made problems. These problems should center on the topics covered in units 1, 2, 3, 4, 5, and 6. Again, at least half of the problems should be in multiple choice form. This will help prepare students for the unit test and the ACT/SAT exams. “Spirals” should be assigned every 2-3 weeks to ensure that students understand and retain important concepts and procedures.

### **Activity-Specific Assessments**

- Activity 1: Solving Right Triangles

There are 36 total measurements in the Solving Right Triangles BLM. Assign 1 point each time a student correctly identifies the given information. Assign 2 points for each calculated measurement. This will result in a total point value of 54. If a smaller point value is preferred, do not assign any points for listing the given information. This will drop the total point value to 36.

Note: Each student should turn in his or her own Solving Right Triangles BLM.

- Activity 2: Applications of Right Triangles

Assign 5 points for each of the three problems. Give 3 points for creating an appropriate real-life situation, 1 point for correctly setting up the problem, and 1 point for solving the problem. Only one BLM per group is turned in, so all three group members will receive the same grade.

- Activity 5: Applications of Oblique Triangles

Remember, only the problems marked with asterisks should be graded. The total point value for the 4 problems will be 24 points. For each selected problem, assign 2 points for a correctly labeled model, 2 points for the appropriate equation(s), and 2 points for generating the correct answer(s).

**Advanced Math – Functions and Statistics**  
**Unit 3: Power Functions and Polynomial Functions**

**Time Frame:** Approximately 4.5 weeks



**Unit Description**

This unit will continue the study of power functions and polynomial functions begun in Algebra II with an approach centered on data collection. Functions that model physical phenomena will be used as examples for a study of the concepts of domain, range, function composition, and function characteristics.

**Student Understandings**

Students will create mathematical models using power and polynomial functions. By analyzing graphical and algebraic models, students will understand that power and polynomial graphs have certain shapes depending on the values of the degrees and coefficients. Students will consider both local and global behavior of power and polynomial functions and how these concepts affect the fit of the model.

**Guiding Questions**

1. Can students recognize and graph power functions?
2. Can students state the domain and range of power functions?
3. Can students describe the end behavior of power functions?
4. Can students identify local and global extrema of power functions?
5. Can students model real-life data using power functions?
6. Can students determine how well their power models fit given or collected data?
7. Can students extrapolate and interpolate values using power models?
8. Can students state limitations of their power models?
9. Can students recognize and graph polynomial functions?
10. Can students state the domain and range of polynomial functions?
11. Can students describe the end behavior of polynomial functions?
12. Can students identify local and global extrema of polynomial functions?
13. Can students model real-life data using polynomial functions?
14. Can students determine how well their polynomial models fit given or collected data?
15. Can students extrapolate and interpolate values using polynomial models?
16. Can students state limitations of their polynomial models?

**Unit 3 Grade-Level Expectations (GLEs)**

<b>GLE #</b>	<b>GLE Text and Benchmarks</b>
<b>Algebra</b>	
4.	Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)
5.	Factor simple quadratic expressions including general trinomials, perfect squares, difference of two squares, and polynomials with common factors (A-2-H)
6.	Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)
7.	Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)
8.	Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)
9.	Solve quadratic equations by factoring, completing the square, using the quadratic formula, and graphing (A-4-H)
10.	Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)
<b>Data Analysis, Probability, and Discrete Math</b>	
19.	Correlate/match data sets or graphs and their representations and classify them as exponential, logarithmic, or polynomial functions (D-2-H)
20.	Interpret and explain, with the use of technology, the regression coefficient and the correlation coefficient for a set of data (D-2-H)
22.	Explain the limitations of predictions based on organized sample sets of data (D-7-H)
<b>Patterns, Relations, and Functions</b>	
24.	Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)
25.	Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)
27.	Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology (P-3-H)
29.	Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)

## Sample Activities

### Activity 1: Know Thyself (GLEs: 6, 7, 8, 25)

Materials List: Know Thyself BLM (one per student), pencils

In this activity, students will rate their own knowledge of mathematical terminology. A *vocabulary self-awareness* chart ([view literacy strategy descriptions](#)) will be completed by the students at this time and continually revised as they progress through Unit 2. Since this chart is designed to allow student to rate his/her own understanding of important terms, no grade should be given.

Students will mark a “+” to indicate that they understand the term well. Students will mark a “√” to indicate limited understanding. Students will mark a “-” to indicate no understanding of the term. The goal is for students to eventually replace all of the minus signs and check marks with plus signs. Because students continually revisit the *vocabulary self-awareness* chart to revise their entries, they have multiple opportunities to practice and extend their understanding of important mathematical terms from both Units 1 and 2.

- Hand out the Know Thyself BLM.
- Explain what the “+”, “√”, and “-” symbols represent.
- Each student will complete the *vocabulary self-awareness* chart on his or her own.
- Do not discuss the results as a class. Students may not be completely honest if they think they will have to share their entries with the rest of the class.
- Explain that students will continually revisit their entries throughout the unit. Let them know that the goal is to replace all of the minus and check signs with plus signs, hopefully, by the end of the unit.

### Activity 2: Power functions – Positive Integer Exponents (GLEs: 4, 6, 7, 8)

Materials List: graphing calculators, Power Functions – Positive Integer Exponents BLM (one per student), pencils

Power functions are defined as  $f(x) = kx^p$ . For this activity,  $p$  will be a positive integer and  $k$  will be one. This activity will help students develop an understanding of the characteristics of power functions (shape, domain, range, end behavior, and local and global extrema) by incorporating the use of a modified *word grid* ([view literacy strategy descriptions](#)).

*Word grids* are utilized to elicit student participation in learning important terms and concepts. Therefore, let the students lead the class discussion about function components and properties. If they leave important items out, use leading questions to guide them in the right direction.

Because these particular power functions are the parent functions (i.e. building blocks) for polynomial functions, it is important to monitor student understanding throughout this activity.

Students should keep the Power Functions – Positive Integer Exponents word grid in their notebooks for future reference. Students should be given time to study the word grid before the unit exam. Keep extra word grids on hand for students that want to complete it again on their own as a study guide.

- Hand out graphing calculators and the Power Functions – Positive Integer Exponents BLM.
- Allow students to form small groups of 2-3.
- Ask the students to start the *word grid* with  $p = 1$ .
- Students should make predictions about the shape of each graph before graphing each function using the calculator.
- Allow the groups to fill out the *word grid* on their own.
- Give the students time to compare answers with other groups.
- Discuss the results as a class.
- Ask the class to predict the graph and characteristics for the following functions:  $f(x) = x^5$ ,  $f(x) = x^6$ ,  $f(x) = x^{17}$ , and  $f(x) = x^{30}$ .
- Be sure to discuss generalizations based on whether the power is even or odd. Examine symmetry characteristics for both even and odd functions. Tie the graphical symmetry to numerical characteristics. For example,  $f(-x) = f(x)$  for an even function and  $f(-x) = -f(x)$  for odd functions. It may be helpful to illustrate these properties using T-charts alongside the graphs.
- Ask students to determine what point all of the power functions have in common and why. *Solution: (0, 1) because any non-zero number raised to the zero power is 1*
- Ask students to order the functions based on their  $y$  values from least to greatest when  $x > 1$ . *Solution:  $f(x) = x$ ,  $f(x) = x^2$ ,  $f(x) = x^3$ ,  $f(x) = x^4$ . It may be helpful to fill in a single T-chart for all four functions in addition to examining all 4 graphs simultaneously.*
- Ask students to order the functions based on their  $y$  values from least to greatest when  $0 < x < 1$ . *Solution:  $f(x) = x^4$ ,  $f(x) = x^3$ ,  $f(x) = x^2$ ,  $f(x) = x$ . Again, it is helpful to fill in a single T-chart for all four functions in addition to examining all 4 graphs simultaneously.*
- Once the students have mastered the graphs and characteristics included in this activity, ask the students to explore parameter changes for  $k$ . For example, what happens to the graph and characteristics when  $k = 2$ ,  $k = 3$ ,  $k = 100$ ,  $k = 1/2$ ,  $k = 1/3$ ,  $k = -1$ ,  $k = -2$ , etc.?

### Activity 3: Modeling with Power Functions – Falling Bodies (GLEs: 4, 7, 8, 10, 24, 25, 29)

Materials List: graphing calculators, TI CBLs or Casio EA100s, motion detectors, meter sticks, various balls (Ping Pong, tennis, racquetball, beach ball, Koosh ball, etc.), masking tape, paper, pencils

In this activity, students will experiment with falling bodies (balls). Students will set up the experiment, collect data by running the experiment, organize the data, and analyze the data. In an ideal situation, the power function would be  $d = 4.9t^2$  if the distance is measured in meters. Actual measurements should be fairly close to this, but do not tell the students. Let them discover this relationship on their own.

NOTE: It is always recommended for teachers to perform the experiment themselves before trying it with students.

- Allow students to form groups of 4-6, depending on the availability of equipment.
- Each group will tape the motion detector to the side of a table. Students should make sure that the motion detector is taped securely and is facing up.
- Each group will plug the motion detector into either the CBL or EA100.
- Each group will connect the CBL or EA100 and the graphing calculator using the link cable.
- Each group will be assigned a certain ball to drop from various heights.
- Each group will run the appropriate “DROP” program.
- Each group will drop its ball from 8-10 different heights. Each drop should happen as soon as the trigger is pressed and should hit the table as close to the motion detector as possible.
- Students will use the meter stick to measure the height of each drop.
- Students will use the graphing calculator to draw the scatterplot from the data transferred to List 1 and List 2.
- Students will explore the fit of various functions by examining the graphs and the coefficient of determination,  $R^2$ . The closer  $R^2$  is to 1, the better the model fits the data.
- Students will select the function that best fits the data and will explain why they chose that particular function.
- Students will identify the independent and dependent variables.
- Students will state the domain and range of the experimental data.
- Students will use their algebraic model to extrapolate and interpolate values.
- Students will state limitations for their algebraic and graphical models.
- Students will share their results with the class. Be sure to discuss whether or not the type of ball affected the outcome.
- Students should also make connections to activities and lessons from physics. For example, the pull of gravity is  $9.8 \text{ m/s}^2$ , half of that is  $4.9 \text{ m/s}^2$ , and the initial velocity is  $0 \text{ m/s}$  since the ball is being dropped not thrown.

Extension: Repeat the experiment, but have students drop various balls from a single height.

Alternate Activity: If you do not have the necessary equipment to run the experiment, give the students the following data to analyze.

Time (sec)	Distance (m)
0	0
.5	1.225
1	4.9
1.5	11.025
2	19.6
2.5	30.625
3	44.1
3.5	60.025

**Activity 4: Polynomial Functions & Their Graphs (GLEs: 4, 6, 7, 8)**

Materials List: graphing calculators, Polynomial Functions & Their Graphs BLM (one per student), paper, pencils

In this activity, students will use technology to analyze polynomial functions. Students will build on knowledge gained from Unit 1 as they complete the Polynomial Functions & Their Graphs BLM. A chart will be used instead of a *word grid* since the labels for the rows and columns will be given to the students. Students will also learn how to identify the multiplicity of zeros in order to determine if graphs will either cross or be tangent to the  $x$ -axis.

- Hand out graphing calculators
- Ask students if they resemble their parents. Let them name particular features/characteristics.
- Tell students that polynomial functions resemble their power function parents. Ask students to predict the power function parent for the polynomial function  $f(x) = (x + 2)(x - 1)^2$ .
- Let students graph the function to determine if their predictions were correct.  
*Solution:  $f(x) = x^3$*
- Ask students to identify the leading coefficient and state its effect on the graph.  
*Solution: The leading coefficient is 1. Since it is positive, the cubic will start at the bottom left and end top right.*
- Tell students that  $x + 2$  and  $x - 1$  are called linear factors since  $f(x) = (x + 2)(x - 1)(x - 1)$ , and each factor has a degree of 1.
- Ask students to find the zeros of the polynomial function. *Solution: -2, 1 (double root)*
- Ask students to describe the behavior of the graph at the zeros. *Solution: The graph crosses the  $x$ -axis at -2 and is tangent to the  $x$ -axis at the double root 1.*
- Ask students to find the extrema and to label them as relative and/or absolute.  
*Solution: Rel. Max. (-1, 4); which is read: the relative maximum is 4 and occurs when  $x$  is -1. Rel. Min. (1, 0); which is read: the relative minimum is 0 and occurs when  $x$  is 1.*
- Ask students to state the increasing and decreasing intervals using interval notation.  
*Solution: increasing  $(-\infty, -1) \cup (1, \infty)$  ; decreasing  $(-1, 1)$*
- Allow students to form groups of 2-4.
- Hand out the Polynomial Functions & Their Graphs BLM.
- Allow each group about 15-20 minutes to work on the BLM.
- Allow groups about 5 minutes to compare answers.
- Discuss the results as a class.

Extension: Groups that finish early should be given polynomial functions in general form. Students will have to work together to factor the functions in order to find the zeros by hand. One example could be  $f(x) = x^3 + 4x^2 - x - 4$ .

**Activity 5: Polynomial Functions & Their Linear Factors (GLEs: 6, 7, 8, 25)**

Materials List: graphing calculators, Polynomial Functions & Their Linear Factors BLM (one copy of each page per student), paper, pencils

The factor theorem and the fundamental theorem of algebra demonstrate that any polynomial of degree  $n \geq 0$  can be expressed as a product of  $n$  linear factors. Thus, linear factors can be viewed as the building blocks of all classes of polynomial functions. This activity will help students make connections among the classes of polynomial functions by examining the graphs of the linear factors. The longer version of this activity was originally published in *The Mathematics Teacher* and can be found at [http://illuminations.nctm.org/print\\_lesson.aspx?id=282](http://illuminations.nctm.org/print_lesson.aspx?id=282).

- Allow students to form groups of 2-4.
- Hand out graphing calculators.
- On the board, graph the line  $y = -2x + 1$  on a coordinate grid.
- Ask each group to find an equation for the line.
- Let volunteers state their equations and how they found them.
- Ask each group to put its equation in slope-intercept form if it is in a different form.
- Ask each group to factor out the slope.
- Let volunteers state their equations and how they found them.  
*Solution:  $y = m(x + b/m)$  and  $y = -2(x - \frac{1}{2})$*
- Ask each group to determine what  $-b/m$  represents by looking at the graph.
- Let volunteers share their answers. *Solution:  $x$ -intercept*
- As a class, let  $c = -b/m$ . The form  $y = m(x - c)$  can then be thought of as the slope/ $x$ -intercept form of a line. Remind students that the factor theorem states that if  $c$  is a root ( $x$ -intercept) of a polynomial function, then  $x - c$  must be a factor of the polynomial function. The only other factor would be  $m$ .
- On the board, graph the line  $y = 3x - 6$ . Use the same coordinate axis as the previous line.
- Repeat the process. *Solution:  $y = 3(x - 2)$  ; 2 is the  $x$ -intercept*
- Ask each group to sketch the function obtained by multiplying the linear factors  $x - \frac{1}{2}$  and  $x - 2$ . The result should be a parabola.
- Let a volunteer come to the board and sketch the parabola.
- Ask each group to interpret the components of the parabola as they relate to the graphs of the lines. *Solution: The lines and the parabola have the same  $x$ -intercepts. The  $y$ -intercept of the parabola is the product of the  $y$ -intercepts of the lines.*
- Tell students to place the ruler or paper strip on the leftmost  $x$ -intercept, so the ruler or paper strip covers the portions of the graphs to the right of that intercept. Ask students to determine if the  $y$  values of the lines (linear factors) still showing are positive or negative. *Solution: The  $y$  values for the line  $y = -2x + 1$  are positive, and the  $y$  values for the line  $y = 3x - 6$  are negative. Since a positive times a negative is negative, the  $y$  values for the parabola are negative on the interval  $(-\infty, \frac{1}{2})$ .*
- Ask students to determine the signs of the  $y$  values for the lines (linear factors) between the  $x$ -intercepts by covering the left and right sides with rulers/paper strips. *Solution: The  $y$  values of the line  $y = -2x + 1$  are negative, and the  $y$  values of the line  $y = 3x - 6$  are positive. Since a negative times a positive is negative, the  $y$  values for the parabola are negative on the interval  $(\frac{1}{2}, 2)$ .*

- Ask students to determine the signs of the  $y$  values for the lines (linear factors) to the right of the rightmost  $x$ -intercept by covering the left side of the graph. *Solution: The  $y$  values of the line  $y = -2x + 1$  are negative and the  $y$  values of the line  $y = 3x - 6$  are positive. Since a negative times a positive is negative, the  $y$  values of the parabola are negative on the interval  $(2, \infty)$ .*
- Assist volunteers in drawing the sign chart for this relationship on the board.

*Solution*

$\frac{-}{-}$	$\frac{0}{+}$	$\frac{+}{-}$	$\frac{0}{0}$	$\frac{-}{+}$	<i>Quadratic Function:</i> $3(x - 2) \bullet -2(x - \frac{1}{2}) = -6(x - 2)(x - \frac{1}{2})$
$\frac{-}{+}$	$\frac{0}{-}$	$\frac{+}{-}$	$\frac{0}{-}$	$\frac{-}{+}$	<i>Linear factor:</i> $3(x - 2) = 3x - 6$
$\frac{+}{\frac{1}{2}}$	$\frac{0}{-}$	$\frac{-}{-}$	$\frac{0}{-}$	$\frac{-}{-}$	<i>Linear factor:</i> $-2(x - \frac{1}{2}) = -2x + 1$

- Ask students the following questions:
  - Is the  $y$  value of the quadratic function positive or negative when  $x = 3$ ? *Positive*
  - Is the  $y$  value of the quadratic function positive or negative when  $x = 0$ ? *Negative*
  - For what values of  $x$  is  $-6(x - 2)(x - \frac{1}{2}) < 0$ ?  $(-\infty, \frac{1}{2}) \cup (2, \infty)$
- Hand out the Polynomial Functions & Their Linear Factors BLMs.
- Group members will work together to complete the BLMs.
- Allow groups to compare answers.
- Discuss the results as a class.

**Extension:** Work Backwards! Give the students the graphs of the linear factors and have them use the  $x$ - and  $y$ -intercepts of the lines to sketch the graph of the polynomial function.

### Activity 6: Polynomial Functions & Their Linear Factors – Part 2 (GLEs: 5, 6, 7, 8, 9, 25)

Materials List: paper, pencils

This activity is a continuation of Activity 5. In this activity, students will algebraically find the linear factors by factoring, quadratic formula, and/or synthetic division. Graphing calculators are not allowed.

- Allow students to form groups of 2-3.
- Ask each group to find the zeros of each function by rewriting each polynomial as the product of its linear factors.
- As volunteers share their results with the class, review Descartes' Rule of Signs (covered in Unit 1) to determine the possible number of positive and negative real roots. Review the Rational Zeros Theorem that states that if  $f$  is a polynomial function with integer coefficients and a degree  $\geq 1$ , the possible rational zeros are  $p/q$  where  $p = \pm$  factors of the last term ( $a_0$ ) and  $q = \pm$  factors of the first term ( $a_n$ ).

- $f(x) = 8x + 12$  *Solution:*  $f(x) = 8(x + 3/2)$  ;  $-3/2$
- $f(x) = 3x^2 - 3x - 18$  *Solution:*  $f(x) = 3(x - 3)(x + 2)$  ;  $3, -2$

3.  $f(x) = 5x^2 + 4x - 3$        $Solution: f(x) = \left(x - \frac{-2 + \sqrt{19}}{5}\right)\left(x - \frac{-2 - \sqrt{19}}{5}\right);$   
 $\frac{-2 \pm \sqrt{19}}{5}$
4.  $f(x) = x^3 + 2x^2 - 4x - 8$        $Solution: f(x) = (x + 2)^2(x - 2); -2$  (double root), 2
5.  $f(x) = x^3 - 6x^2 + 11x - 6$        $Solution: f(x) = (x - 1)(x - 2)(x - 3); 1, 2, 3$
6.  $f(x) = x^4 - x^3 - 5x^2 + 3x + 6$        $Solution: f(x) = (x - 2)(x + 1)(x - \sqrt{3})(x + \sqrt{3});$   
 $2, -1, \pm\sqrt{3}$

- Ask each group to work backwards by creating zeros for function and then multiplying out the linear factors to find the polynomial. Each group should create one linear, two quadratic, and two cubic polynomial functions.
- Each group will keep one copy of the newly created polynomial functions and swap another copy with a different group. The copy that is swapped should only contain the polynomial function, not the linear factors.
- For homework, students will use algebraic methods (factor, quadratic formula, and/or synthetic division) to rewrite the polynomial functions as the product of linear factors and will then find the zeros.
- The next day, students will give their problems back to the creators so that they make check their work. Give the students 5-10 minutes to check the answers before they return them to their owners.

**Extension:** Turn the problems 1-6 into polynomial inequalities. Make sure to use all four inequalities: greater than 0, less than 0, greater than or equal to 0, and less than or equal to 0. Students should construct sign charts to justify their answers.

### Activity 7: Applications of Polynomial Functions (GLEs: 4, 6, 8, 10, 24, 25, 27, 29)

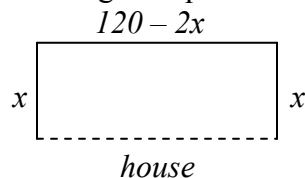
Materials List: graphing calculators, Applications of Polynomial Functions I BLM (one per student), 42 in x 48 in poster boards (one per group), scissors (one per group), tape, yardsticks (one per group), Applications of Polynomial Functions II BLM (one per student), paper, pencils

In this activity, students will create algebraic and graphic models for real-life situations. Students will use these models to solve problems.

#### Situation #1

- Allow students to form groups of 3-4.
- Hand out graphing calculators.
- Present students with the following situation: Hannah has 120 meters of fencing to create a rectangular pen for her pet rabbit, Hoppy. If she uses the back of her house as one side of the rectangular pen, what dimensions will maximize the pen's area? What is the maximum area?

- Do not give the students any hints. Let them work together for about 10-15 minutes. Then, let students share their approaches to the problem.
- Let volunteers share their diagrams/pictures of the pen. Make sure that all three sides are labeled. *Solution:*



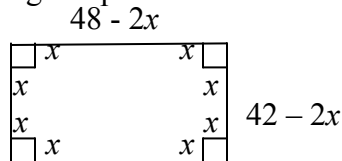
- Ask each group to write an algebraic model for the area of the pen. *Solution:  $A = x(120-2x)$  or  $A = 120x - x^2$  function.*
- Ask students to predict the shape of the area
- Tell students to graph the area function. *Solution: The parabola should open downward and cross the  $x$ -axis at 0 and 60.*
- Ask students to use the graph to find the dimensions and the maximum area. Give them about 10 minutes to work with their group members.
- Discuss the results as a class. *Solution:  $x$  is 30 m and is found by using the maximum feature on the graphing calculator. Since the function is quadratic in nature, the maximum ( $x$  coordinate of the vertex) could be found by hand by using the formula  $V_x = -b/2a$  or by finding the midpoint of the  $x$ -intercepts. Substituting 30 for  $x$  in  $120-2x$  gives the other dimension, 60 m. Thus, the dimensions that will maximize the area of the rectangular pen are 30m by 60m. This gives a maximum area of 1,800 square meters.*
- Discuss limitations of the graphical and algebraic models. Hopefully, students will notice that the  $x$  dimension can only be so large in order for the third side of the pen to exist. Specifically,  $0 < x < 60$  m.
- Hand out the Applications of Polynomial Functions I BLM.
- Let each group work on its own for about 10-15 minutes.
- Let the groups compare answers.
- Discuss the results as a class. Hopefully, students will notice that one dimension is always twice the other dimension for this type of situation.
- Discuss limitations of the graphical and algebraic models. Hopefully, students will notice that the  $x$  dimension can only be so large in order for the third side of the pen to exist. Specifically,  $0 < x < 400$  m.

## Situation #2

- Allow students to form groups of 2-4.
- Hand out graphing calculators.
- Present students with the following problem: A box, without a top, is formed by removing squares of equal size from each corner from the poster board. What dimensions will maximize the volume? What is the maximum volume?
- Tell each group to arbitrarily select the size of the squares that will be cut out from each corner of the poster board. Let students cut out the squares and then fold up the sides to form a box. If necessary, tape can be used to secure the sides of the box.

- Tell each group to find the volume of its box. Let students record their results on the board. Circle the largest volume and ask that group to bring its box to the front of the room. Tell students that mathematics will determine whether that is the best possible box, in terms of volume, that could be made from the poster board.
- Hold up a poster board for students to see. Ask a volunteer to come up and measure the dimensions of the poster board. Write the dimensions on the board.
- Ask each group to draw a diagram/picture of this situation. Tell students to label the squares as  $x$ .

➤ *Solution:*



- Ask each group to write an algebraic model to represent the volume of the box when the sides are folded up. Let volunteers write their models on the board.

*Solution:*  $V = x(48 - 2x)(42 - 2x)$

- Ask each group to graph the volume function. *Solution: The cubic function will cross the  $x$ -axis at 0, 24, and 21.*
- Ask each group to find the size of the squares that should be cut from each corner in order to maximize the volume. Also, ask students to find the dimensions that will maximize the volume and the maximum volume.
- After about 10-15 minutes, let groups compare answers.
- Discuss the results as a class. Be sure to discuss which part of the cubic graph is appropriate for this situation. *Solution: Square sides  $\approx 7.45$  in  
Dimensions  $\approx 7.45$  in by 33.10 in by 27.10 in      Max. Volume  $\approx 6,682.72$  in<sup>3</sup>*
- Discuss limitations of the algebraic and graphical models. Be sure that students realize that the size of the squares cut out can only be so big if a box is to be formed. Algebraically, it is impossible for  $x$  to be greater than or equal to 21 in. Larger values would result in a negative length or width, which is not possible. Graphically, the same relationship can be seen as the  $y$  values (volume) become negative when  $21 < x < 24$ .
- Hand out the Applications of Polynomials II BLM.
- Each group works on its own and turns in the results for a grade. See Sample Assessments for guidelines.

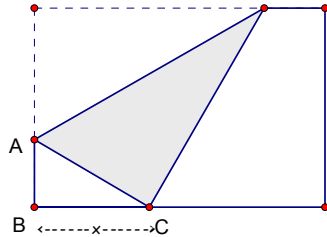
### Activity 8: Data and Polynomials (GLEs: 4, 7, 8, 10, 19, 20, 22, 24, 25, 29)

Materials List: graphing calculators, unlined paper 8.5 in by 11 in (one per group), rulers, paper, pencils

In this activity, students will record the area of triangles formed by folding one corner of the paper to meet the other side of the paper. Students will find algebraic and graphical models for the area of the triangles based on the length of side  $x$ . Students will use those models to find the triangle with maximum area.

- Allow students to form groups of 2-3.
- Hand out graphing calculators.

- Demonstrate how to fold the paper. Make sure that students understand that the length of side  $x$  can and should vary from group to group. See diagram below.



- Ask each group to measure the length of  $x$  in centimeters.
- Ask each group to find the area of its triangle. If students do not know how to find the area on their own, tell them to measure sides  $AB$  and  $BC$  and then use the area formula  $A = \frac{1}{2}bh$ .
- Draw a chart on the board and let each group fill in its data. Make sure that there are both large and small values of  $x$ . If not, ask a couple of groups to make additional measurements to ensure a wide range of values for  $x$ .

Length of side $x$ (cm)								
Area of triangle (cm <sup>2</sup> )								

- Ask each group to state domain restrictions for this situation. Let volunteers share their domain restrictions with the class. *Solution:  $0 < x < 28$  cm (width of the paper)*
- Ask students to enter the data into List 1 and List 2 on the graphing calculator.
- Ask students to graph the scatterplot on the graphing calculator.
- Ask each group to find the polynomial model that best fits the data. Let volunteers present their models and tell why they selected that particular polynomial model. Discussions should include the coefficient of determination,  $R^2$ . This number,  $0 \leq R^2 \leq 1$ , represents the percent of the data that is accounted for by curve of best fit. The closer to 1, the better the fit. The coefficient of determination is used for nonlinear models. *Solution: The model of best fit will vary based on the data collected in class. Select the polynomial model with the largest  $R^2$  value.*
- Ask each group to use the model of best fit to find the  $x$  value for the triangle with maximum area. Also, ask each group to find the maximum area.
- Let volunteers share their results with the class.
- Discuss the results as a class. *Solution: Both values will vary based on the model chosen. The value of  $x$  and the maximum area will be found by using the maximum feature on the graphing calculator.*

## Sample Assessments

### General Assessments

- Each student will create a second entry for his or her *learning log* ([view literacy strategy descriptions](#)). Remember, students will be adding entries to the *learning logs* throughout the school year. The second entry will have the title, “How to Find the Zeros of Polynomial Functions.” This assessment piece can be formal or informal in nature. Informally, students would be assigned a grade for writing the entry. This will let the teacher know what students do and do not understand about finding the zeros of polynomial functions. Formally, students can be graded based on the inclusion of main topics like  $x$ -intercepts, linear factors, multiplicity of roots, graphic methods, algebraic methods, Descartes’ Rule of Signs, and the Rational Zeros Theorem.
- Each student will continue to add terms from this unit to the glossary started in Unit 1. This glossary will be included in a student portfolio that will be graded at the end of each semester.
- Each student will complete a “spiral” comprised of 5 -10 teacher-made problems. These problems should center on the topics covered in both Units 1 and 2. Again, at least half of the problems should be in multiple choice form. This will help prepare students for the unit test and the ACT/SAT exams. “Spirals” should be assigned every 2-3 weeks to ensure that students understand and retain important concepts and procedures.

### Activity-Specific Assessments

- Activity 3: Modeling with Power Functions – Falling Bodies

If students are well versed with labs, Activity 3 (Falling Bodies) could be assigned for a grade. Each student or each group should create and turn in a lab report for the experiment. Possible headings for the lab report include: title, description, prediction (of what will happen to the time it takes for the ball to hit the table as the height increases), procedure, data, results, and conclusion. While rubrics and point totals are always left to the teacher’s discretion, a sample rubric is provided below.

Points	Missing (0 pts)	Poor (1 pt)	Lacking (2 pts)	Adequate (3 pts)
<b>Title</b>	Missing	Included	-----	-----
<b>Description</b>	Missing	Missing several main ideas	Missing one idea	Includes all main ideas
<b>Prediction</b>	Missing	Included but unclear	Included and clear	-----
<b>Procedure</b>	Missing	Missing several steps or inaccurate steps	Missing one step or partially inaccurate	Includes all steps and is accurate (set up, starting distance, distance increments)

<b>Data</b>	Missing	Fewer than eight measurements	Includes 8-10 measurements	-----
<b>Results</b>	Missing	Inaccurate power model, missing or poor explanation of why that model was chosen	Accurate power model & poor explanation of why that model was chosen	Accurate power model & accurate explanation of why that model was chosen (shape and $R^2$ )
<b>Conclusion</b>	Missing	Describes the direct relationship but does not address limitations of the model	Describes the direct relationship but inaccurately addresses limitations of the model or vice versa	Accurately describes both the direct relationship and limitations of the model

➤ Activities 5 and 6: Polynomial Functions and Their Linear Factors

Allow students to pick a partner. Hand out graphing calculators. Place the polynomial function  $f(x) = -x^4 - 3x^3 + 4x^2$  on the board. Students must rewrite the polynomial function as a product of its linear factors and then state the zeros. If multiplicity exists, students must identify it. Students will raise their hands when they are finished with the problem. The teacher will take their graphing calculators and show them the next polynomial function  $f(x) = x^3 + 4x^2 - 3x - 18$ . This function should be written on a piece of paper instead of the board in order to ensure that students work the problem by hand. This time, students must use algebraic methods (factoring, quadratic formula, and/or synthetic division) to rewrite the polynomial function as the product of its linear factors and then find the zeros. Multiplicity must be addressed if it exists. Partners may work together, but each student needs to turn in his or her own work.

$$\begin{array}{llll} \text{Solutions: } f(x) = -x^4 - 3x^3 + 4x^2 & f(x) = -x \bullet x(x-1)(x+4) & 0 \text{ (double root), } 1, -4 \\ f(x) = x^3 + 4x^2 - 3x - 18 & f(x) = (x+3)(x+3)(x-2) & -3 \text{ (double root), } 2 \end{array}$$

➤ Activity 7: Applications of Polynomial Functions

Ask students to turn in the Applications of Polynomial Functions II BLM. Each group member should turn in his or her own paper. Check the answers using the Applications of Polynomials Functions II with Answers BLM.

Scoring guidelines are left to the teacher's discretion. One possible way to score the BLM is to assign 5 points to each component (diagram, algebraic model, graphic model, limitations of the models, and the solution).

**Advanced Math – Functions and Statistics**  
**Unit 4: Power Functions and Rational Functions**

**Time Frame:** 3.5 weeks



**Unit Description**

This unit focuses on rational and radical functions. Relationships to their power function parents will be explored. Each type of function will be analyzed based on of its algebraic and graphic characteristics and properties.

**Student Understandings**

Students must be able to distinguish between rational and polynomial functions from verbal descriptions, graphs, and algebraic properties. Students will identify the local and global characteristics of rational functions and use this knowledge to sketch their graphs, with and without technology. Students will also model and solve real-life problems using rational functions.

**Guiding Questions**

1. Can students recognize and graph power functions?
2. Can students state the domain and range of power functions?
3. Can students describe the end behavior of power functions?
4. Can students identify local and global extrema of power functions?
5. Can students model real-life data using power functions?
6. Can students determine how well their power models fit given or collected data?
7. Can students extrapolate and interpolate values using power models?
8. Can students state limitations of their power models?
9. Can students identify rational functions given an equation?
10. Can students identify rational functions given a graph?
11. Can students state the domain and range of rational functions?
12. Can students sketch and analyze graphs of rational functions based on their characteristics (asymptotes, discontinuity, zeros, local and global extrema, increasing/decreasing intervals, end behavior, and symmetry)?
13. Can students solve rational equations?
14. Can students model real-life problems and data using rational functions?
15. Can students determine the fit of a rational model?
16. Can students use rational models to interpolate and extrapolate data?

## Unit 4 Grade-Level Expectations (GLEs)

GLE #	GLE Text and Benchmarks
<b>Algebra</b>	
4.	Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)
5.	Factor simple quadratic expressions including general trinomials, perfect squares, difference of two squares, and polynomials with common factors (A-2-H)
6.	Analyze functions based on zeros, asymptotes, and local and global characteristics for the function (A-3-H)
7.	Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions. (A-3-H)
8.	Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)
10.	Model and solve problems involving quadratic, polynomial, exponential logarithmic, step function, rational, and absolute value equations using technology (A-4-H)
<b>Data Analysis, Probability, and Discrete Math</b>	
19.	Correlate/match data sets or graphs and their representations and classify them as exponential, logarithmic, or polynomial functions (D-2-H)
<b>Patterns, Relations, and Functions</b>	
20.	Interpret and explain, with the use of technology, the regression coefficient and the correlation coefficient for a set of data (D-2-H)
24.	Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)
25.	Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)
27.	Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology. (P-3-H)
29.	Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)

## Sample Activities

### Activity 1: Power Functions – Negative Integer Exponents (GLEs: 4, 6, 7, 8)

Materials List: graphing calculators, Power Functions – Negative Integer Exponents BLM (one copy per student), pencils

Power functions with negative integer exponents are parent functions for rational functions. Thus, it is important to make sure that students develop a strong foundation for power functions before moving on to subsequent activities. This activity will utilize a modified *word grid* ([view literacy strategy descriptions](#)) to help students further their understanding of power functions. A modified *word grid* will be used since the labels for the first row are given. Once again, let  $k$  equal one. The class should determine important function components and characteristics to list in the first column.

*Word grids* are utilized to elicit student participation in learning important terms and concepts. Therefore, let the students lead the class discussion about function components and characteristics. If they leave important items out, use leading questions to guide them in the right direction.

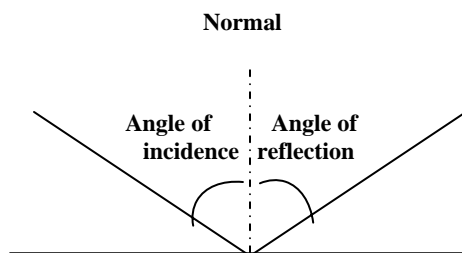
Students should keep the Power Functions – Negative Integer Exponents word grids in their notebooks for future reference. Students should be given time to study the word grid before the unit exam. Keep extra word grids on hand for students that want to complete it again on their own as a study guide.

- Hand out graphing calculators and the Power Functions – Negative Integer Exponents BLM
- Allow students to form small groups of 2-3.
- Ask the students to start the *word grid* with  $p = -1$  and continue to  $p = -4$  for the Negative Exponents BLM.
- Students should make predictions about the shape of each graph before graphing each function using the calculator.
- Allow the groups to fill out the *word grid* on their own.
- Give the students time to compare answers with other groups.
- Discuss the results as a class.
- Ask the class to predict the graph and characteristics for  $f(x) = x^{-6}$  and  $f(x) = x^{-7}$ .
- Be sure to discuss generalizations based on whether the power is even or odd. Examine symmetry characteristics for both even and odd functions. Tie the graphical symmetry to numerical characteristics. For example,  $f(-x) = f(x)$  for an even function and  $f(-x) = -f(x)$  for odd functions.
- Once you feel that the students have mastered the graphs and characteristics included in this activity, ask the students to explore parameter changes for  $k$ . For example, what happens to the graph and characteristics when  $k = 2$ ,  $k = 3$ ,  $k = 100$ ,  $k = 1/2$ ,  $k = 1/3$ ,  $k = -1$ ,  $k = -2$ , etc.?

## Activity 2: Modeling with Power Functions–Law of Reflection (GLEs: 4, 7, 8, 10, 20, 24, 29)

Materials List: graphing calculators, meter sticks, long strips of paper, washable markers, small mirrors, paper, pencils

In this activity, students will conduct an experiment regarding the Law of Reflection. The Law of Reflection states that when light strikes a surface, the angle of incidence is equal to the angle of reflection.



NOTE: It is always recommended for teachers to perform the experiment themselves before trying it with students. Additional details can be found at the following Texas Instruments web site: <http://education.ti.com/educationportal/activityexchange/Activity.do?cid=US&aId=4259>.

- Allow students to form groups of 3.
- Allow students to explore the Law of Reflection by holding a mirror in the palm of their hands parallel to the floor and at a level that allows the students to look into the mirror. Ask students to find an object that is higher than eye level and observe what happens to the reflection in the mirror as they walk towards and away from the object.
- Each group tapes a long strip of paper to a wall.
- Each group uses a meter stick to measure and label the following heights from the floor: 10 cm, 20 cm, 30 cm, and so on up to 100 cm. It is helpful if the numbers written on the paper strip are at least 3 cm tall.
- Each group will tape a second, and longer, strip of paper on the floor perpendicular to the first strip. Students should make sure that the strips of paper meet at zero.
- On the second strip of paper, each group measures and labels the following lengths starting from zero: 10 cm, 20 cm, 30 cm, and so on up to 350 cm. Again, it is helpful if the numbers are at least 3 cm tall.
- Using a washable marker, each group will draw a 2 cm x 2 cm square in the center of the mirror. Each group should place the mirror on the floor so that the center of the square is 20 cm from the wall.
- Each group will select one member to be the spotter, one the marker, and one the scribe.
- Each group will measure the eye-level height of the spotter using a meter stick.
- The spotter will face the wall. Ask the spotter to move away from the wall until the reflection of the numeral 10 can be seen in the mirror. The marker will mark the position of the spotter's toe on the paper strip on the floor. The scribe will record the distance.
- Ask the spotter to walk slowly towards the wall until the reflection of the numeral 20 can be seen in the mirror. Again, the marker will mark the position of the spotter's toe on the paper strip on the floor and the scribe will record the distance.
- Each group repeats the process until all of the numerals on the wall have been spotted.
- Students may want to swap jobs so that each member gets a chance to be the spotter.
- Students will enter the heights along the wall into List 1 using the graphing calculator.

- Students will enter the lengths along the floor into List 2 using the graphing calculator.
- Students will use the graphing calculator to draw the scatterplot.
- Students will find the power function that models the data. The coefficient of determination,  $R^2$ , should be used to examine the fit of the power function. The closer  $R^2$  is to 1, the better the model fits the data.
- Students will identify the independent and dependent variables.
- Students will state the domain and range of the experimental data.
- Students will use their algebraic model to extrapolate and interpolate values.
- Students will state limitations for their algebraic and graphical models.
- Results of the experiment can either be shared as a class or turned in for a grade. See sample assessments for a possible grading rubric.

### Activity 3: Graphs of Rational Functions – Part One (GLEs: 4, 6, 7, 8, 19, 27)

Materials List: graphing calculators, paper, pencils

In this activity, students will work with improper functions that appear rational at first glance. After factoring each function, students will be left with a linear function that is discontinuous at a certain point. The degree of the numerator is one larger than the degree of the denominator for improper rational functions. The first part of the activity focuses on rational functions that reduce, and thus leave a hole in the graph. The second part of this activity will explore rational functions that do not reduce, which produces an oblique or slant asymptote.

- Write the function  $f(x) = \frac{x^2 - 1}{x + 1}$  on the board. Ask students to predict the power function power and the asymptotes for this function.
- Ask students to graph the function and revise their answers. Discuss the results.  
*Solution: The power function parent is  $f(x) = x$ . There are no asymptotes because the function reduces to  $f(x) = x - 1$  after it is factored.*
- The factored function is linear. However, there is a hole in the graph at  $x = -1$  because the original function is not defined at the point  $(-1, -2)$ .
- Ask students to identify the domain and range of the function.  
*Solution: Domain  $(-\infty, -1) \cup (-1, \infty)$  Range  $(-\infty, -2) \cup (-2, \infty)$*
- Ask students to find the zero of the function both graphically and algebraically.  
*Solution: 1*
- Discuss the discontinuity of the function. Since there is a hole in the graph at  $x = -1$ , the function is discontinuous at  $x = -1$ . More specifically, the function demonstrates point or removable discontinuity at  $x = -1$ .
- Make sure the students sketch the graph of the function by using the asymptotes and identify one point on each branch by selecting one  $x$  value to the right of the vertical asymptote and one to the left.
- Repeat the process for  $f(x) = \frac{x^2 - x - 6}{x - 3}$ .

The next part of the activity moves on to rational functions that have an oblique/slant asymptote.

- Write the function  $f(x) = \frac{x^2 - 4}{x - 1}$  on the board. Ask students to predict the power function parent and the asymptotes for the function.
- Tell the students to graph the function using the graphing calculator. Give the students time to revise their predictions.
- Discuss the results. *Solution: Parent  $f(x) = x$  Vertical asymptote  $x = 1$*
- Students will notice from the graph that there is no horizontal asymptote. Some may notice that there is a diagonal asymptote. Ask students if the diagonal asymptote has a positive or negative slope. *Solution: positive slope*
- Tell the students that the diagonal (oblique) asymptote is  $y = x + 1$ . Ask students to work together to find an algebraic method for finding the equation of the oblique asymptote.
- Let volunteers share their methods with the class. Discuss the results.  
*Solution: Divide  $x^2 - 4$  by  $x - 1$  and drop the remainder. Work this out on the board.*
- Ask the students to state the domain and range of the function.  
*Solution: Domain  $(-\infty, 1) \cup (1, \infty)$  Range  $(-\infty, \infty)$*
- Ask students to find the zeros of the function both algebraically and graphically.  
*Solution: 2, -2*
- Discuss the discontinuity of the function. *Solution: Infinite discontinuity exists at  $x = 1$ .*
- Make sure that the students sketch the graph of the function using the asymptotes and by determining one point on each branch by selecting an  $x$  value to the right and left of the vertical asymptote.
- Repeat the process for  $f(x) = \frac{2x^2 - 5}{x + 2}$ .

Extension: Repeat the process with functions whose degree of the numerator is two or more than the degree of the denominator. One example would be  $f(x) = \frac{x^4 - 3x^2 - 4}{x^2 - 1}$ .

#### Activity 4: Graphs of Rational Functions – Part Two (GLEs: 4, 6, 7, 8, 27)

Materials List: graphing calculators, paper, pencils

"Rational function" is the name given to a function that can be represented as the quotient of polynomials, just as a rational number is a number that can be expressed as a quotient of whole numbers. The algebraic representation of a rational function is  $f(x) = p(x)/q(x)$ .

- Ask the class the following question, "What are the differences between rational functions and polynomial functions?" It may be helpful to write an example of each type of function on the board and let the students graph them. *Possible Answers: A rational function is the quotient of two polynomial functions. Polynomial functions are continuous. Rational functions have asymptotes and are discontinuous. The domain for all polynomial functions is  $(-\infty, \infty)$ . The domain of rational functions is dependent upon the denominator.*

- Write the function  $f(x) = \frac{1}{x-2}$  on the board. Ask the students if this is a polynomial or rational function and why.
- Ask students to identify the parent function. *Solution:  $f(x) = x^{-1}$  or  $f(x) = 1/x$*
- Ask students to name the shape of rational functions. *Solution: hyperbola*
- Ask students to predict the translation caused by subtracting 2 from  $x$ .
- Let students graph the function using the graphing calculator. Discuss the translation by comparing this graph to its parent graph. *Solution: 2 units right*
- Ask students to state the domain of the function. Let volunteers write their answers on the board.
- Discuss the results. *Solution:  $(-\infty, 2) \cup (2, \infty)$  Be sure that students recognize that substituting 2 for  $x$  results in an undefined answer. Therefore, 2 cannot be part of the domain. Graphically, the function extends forever left and forever right but jumps over the asymptote at  $x = 2$ .*
- Ask students to find the range of the function. Let volunteers write their answers on the board. *Solution:  $(-\infty, 0) \cup (0, \infty)$  Algebraically, 1 divided by any non-zero number cannot result in an answer of 0. Graphically, the function goes down forever and up forever but jumps over the horizontal asymptote  $y = 0$ .*
- Looking at the graph, ask students to state the horizontal and vertical asymptotes. *Solution: Vertical Asymptote  $x = 2$  Horizontal Asymptote  $y = 0$*
- Discuss the discontinuity of the function. *Solution: Since the function has an asymptote at  $x = 2$ , it is discontinuous at this value. More specifically, the graph demonstrates infinite or non-removable discontinuity at  $x = 2$ .*
- Ask students to find an algebraic method to find the horizontal asymptote. Let them discuss this problem in groups of 2-3. Let volunteers share their methods with the class. *Solution: To find the vertical asymptote, set the denominator equal to 0. To find the horizontal asymptote, find the term with the largest degree in the numerator and in the denominator. Then, divide the terms.*
- Ask students to find one point for each branch of the hyperbola by selecting an  $x$  value to the right and the left of the vertical asymptote. *Answers will vary.*
- Ask students if the hyperbola has a maximum or a minimum. *Solution: No. There is no largest or smallest  $y$  value because  $y \rightarrow \infty$  on the right branch and  $y \rightarrow -\infty$  on the left branch.*
- Ask students to find the zeros of the function. *Solution: There are no zeros since there are no  $x$ -intercepts.*
- Ask students to find the intervals where the function is increasing and decreasing. Let volunteers share their answers with the class. *Solution: Increasing: None  
Decreasing:  $(-\infty, 2) \cup (2, \infty)$*

Repeat the steps above for the following functions:

$$2. f(x) = \frac{-1}{x+1} \quad 3. f(x) = \frac{3}{2x-4} \quad 4. f(x) = \frac{2x}{x+5} \quad 5. f(x) = \frac{x+1}{x^2-1}$$

**Extension:** For students demonstrating good backgrounds with rational functions from algebra II, let them move on to rational functions with two vertical asymptotes like

$$f(x) = \frac{x-1}{x^2-4}.$$

### Activity 5: Solving Rational Inequalities (GLEs: 5, 6, 8)

Materials List: graphing calculators, paper, pencils

In the first two activities, the zeros of rational functions of the form  $f(x) = p(x)/q(x)$  were easy to find. After setting  $f(x)$  equal to 0, we multiplied both sides of the equation by  $q(x)$ . The result was the same as solving the polynomial equation  $p(x) = 0$ , i.e. finding the roots of  $p(x)$ .

In this activity, students will use the zeros of both the numerator and the denominator in order to determine where the function is greater than 0, greater than or equal to 0, less than 0, or less than or equal to 0.

- Ask students to state the different ways a rational function can be greater than zero.

*Solution:*  $+/+$  or  $-/-$

- Ask students to state the different ways a rational function can be less than zero.

*Solution:*  $-/+$  or  $+/-$

- Ask students what separates positive numbers from negative numbers.

*Solution:* 0

- Ask students to find the zeros for the numerator and denominator for  $f(x) = \frac{x-4}{x+3}$ .

*Solution:* Numerator  $x = 4$  Denominator  $= -3$

- Ask students to draw a sign chart for the function.

$+$	$0$	$-$	$0$	$+$	<i>Rational function:</i> $x - 4/x + 3$
$-$	$-$	$0$	$+$	$+$	<i>Linear factor:</i> $x - 4$
$-$	$0$	$+$	$+$	$+$	<i>Linear factor:</i> $x + 3$
$-3$	$4$				

- Ask students to use the sign chart to solve each inequality.

a)  $\frac{x-4}{x+3} \leq 0$     b)  $\frac{x-4}{x+3} < 0$     c)  $\frac{x-4}{x+3} \geq 0$     d)  $\frac{x-4}{x+3} > 0$

- Before discussing the results, ask students to check their solutions by looking at the graph of the function.

*Solutions:* a)  $(-3, 4]$     b)  $(-3, 4)$     c)  $(-\infty, -3) \cup [4, \infty)$     d)  $(-\infty, -3) \cup (4, \infty)$

- Allow students to form groups of 2-4.

- Ask students to solve the following inequalities:

1. $\frac{x+5}{x+1} > 0$	2. $\frac{(x-2)^2}{x-9} \leq 0$	3. $\frac{x^2-16}{2x+12} < 0$
4. $x + \frac{12}{x} \geq 7$	5. $\frac{3x-4}{x+6} < 2$	6. $\frac{5}{x-2} \leq \frac{2}{x+1}$

- Remind students to check their answers by looking at the graph of each function.
- Let volunteers share their results with the class.  
*Solutions:* 1.  $(-\infty, -5) \cup (-1, \infty)$     2.  $(-\infty, 9]$     3.  $(-\infty, -6) \cup (-4, 4)$   
                   4.  $(-\infty, 3] \cup [4, \infty)$     5.  $(-6, 10)$     6.  $(-\infty, -3] \cup (-1, 2)$
- Allow groups to create 3 problems and swap them with another group. Students should work the swapped problems for homework.

Extension: Give students inequalities with more than three factors to solve. One example would be  $\frac{x(x+2)(3-x)}{(x+7)(x-4)} > 0$ .

### Activity 6: Applications of Rational Functions (GLEs: 5, 6, 8, 10, 25, 27, 29)

Materials List: graphing calculators, Applications of Rational Functions BLM (one per group), pencils

This activity will incorporate the use of modified *story chains* ([view literacy strategy descriptions](#)). A story chain involves a small group of students writing a story problem. The first student starts the story. The next student adds a line and passes it to the next student to do the same. For this modified version of a *story chain*, each group will be given real-life situations. Group members will take turns creating questions for the group to answer based on the real-life situations.

- Allow students to form groups of 3.
- Hand out graphing calculators to all students.
- Write the following real-life situation on the board: The temperature of food placed in a refrigerator is given by  $T = 10 \left( \frac{4t^2 + 16t + 75}{t^2 + 4t + 10} \right)$ , where  $T$  is the temperature of the food in degrees Fahrenheit and  $t$  is the amount of time the food has been in the refrigerator in hours.
- Give students a couple of minutes to generate questions based on the real-life situation.
- Ask volunteers to share their questions with the class.
- Give the class time to answer the questions to the satisfaction of the authors.
- If students struggle to create meaningful questions, either lead them to or provide the following examples.

What is the initial temperature of the food?     $T = 75^\circ\text{F}$

What will the temperature of the food be 2 hours after being placed in the refrigerator?

$T \approx 55.9^\circ\text{F}$

When will the temperature of the food be  $45^\circ\text{F}$ ?     $t = 6 \text{ hours}$

Once placed in the refrigerator, the food will never drop below what temperature?

$T = 40^\circ\text{F}$  (because this is the horizontal asymptote)

- Hand out the Applications of Rational Functions BLMs (one per group).

- Explain the modified *story chain* to the students.
- Group members will take turns creating and answering questions for each of the three real-life situations. The first group member will write the first question based on the first real-life situation. The other two group members will answer the question. The first member will check the answer and provide feedback to the rest of the group.
- Once the first question has been answered correctly, the second group member will write a question for the rest of the group to answer based on the same real-life situation. The second member will check the answer and provide feedback to the rest of the group.
- Once the second question has been answered correctly, the third group member will write a question for the rest of the group to answer based on the first real-life situation. The third member will check the answer and provide feedback to the rest of the group.
- Once the third question has been answered correctly, the process is repeated for the second real-life situation and then again for the third real-life situation.
- Each group can turn in the Applications of Rational Functions BLM for a grade or groups can swap and check each other's papers.
- Sample questions are provided in the Applications of Rational Functions with Answers BLM. The sample questions could be answered by the class at the end of the activity as a summary or wrap-up of applications of rational functions.

### **Activity 7: Mathematical Masterminds (GLEs: 4, 6, 7, 8, 10, 20, 24, 25, 27, 29)**

Materials List: graphing calculators, paper, pencils, professor attire – optional (ties, pocket protectors, crowns, tiaras, thinking caps, etc.)

Mathematical masterminds is a spin-off of the literacy strategy *professor know-it-all* ([view literacy strategy descriptions](#)). This strategy is used when coverage of new content for the unit has been completed. It also provides a fun way for students to monitor and demonstrate their own understanding of the material covered throughout the unit.

- Allow students to form groups of 3-4.
- Ask students to review all material from this unit (Activities 1-6).
- Give each group about 10-15 minutes to create 3-5 questions to ask the mathematical masterminds.
- Ask one group to come to the front of the room. Tell the class that the students standing before them are mathematical masterminds. Their job is to answer questions posed by the class.
- Allow a student to ask a question.
- Tell the mathematical masterminds to huddle together and come up with an answer. One of the masterminds should share the answer with the class.
- The class either agrees with the solution or challenges the mathematical masterminds to defend and/or revise their answer.
- Once the class and the teacher are satisfied with the answer, another student asks a question and the process repeats itself.
- Assign a time and/or question limit so that several groups get a chance to come up to the front of the room.

- If the class is leaving out important content questions, lead them towards the questions that need to be asked or ask a question yourself.

## Sample Assessments

### General Assessments

- Each student will create a third entry for his or her *learning log* ([view literacy strategy descriptions](#)). Remember, students will be adding entries to the *learning logs* throughout the school year. The third entry will have the title, “How to Graph a Rational Function.” This assessment piece can be formal or informal in nature. Informally, students would be assigned a grade for writing the entry. This will let the teacher know what students do and do not understand about graphing rational functions. Formally, students can be graded based on the inclusion of main topics like discontinuity,  $x$ -intercepts,  $y$ -intercept, vertical asymptotes, horizontal asymptotes, oblique/slant asymptotes, domain, range, etc.
- Each student will continue to add terms from this unit to the glossary started in unit one. This glossary will be included in a student portfolio that will be graded at the end of each semester.
- Each student will complete a “spiral” comprised of 5 -10 teacher-made problems. These problems should center on the topics covered in Units 1, 2 and 3. Again, at least half of the problems should be in multiple choice form. This will help prepare students for the unit test and the ACT/SAT exam. “Spirals” should be assigned every 2-3 weeks to ensure that students understand and retain important concepts and procedures.

### Activity-Specific Assessments

- Activity 1: Power Functions – Negative Integer Exponents

Since students are familiar with word grids from Unit 2, let them turn in the Power Functions – Negative Integer Exponents BLM for a grade. For example, students could be awarded two points for each correct entry, 1 point for each incorrect entry, and 0 points for no entry. This would make the assignment worth 36 points. Leave the point total at 36 or make it 30-35 and allow for bonus points. Use the Power Functions – Negative Exponents BLM Answer Key to award point values.

- Activity 2: Modeling with Power Functions – Law of Reflection

If students performed well on the Falling Bodies Activity from Unit 2, this activity could be assigned for a grade. Each student or each group should create and turn in a lab report for the experiment. Possible headings for the lab report include: title, description, prediction (what will happen to the intensity as the distance increases), procedure, data, results, and conclusion. While

rubrics and point totals are always left to the teacher's discretion, a sample rubric is provided below.

<b>Points</b>	<b>Missing (0 pts)</b>	<b>Poor (1 pt)</b>	<b>Lacking (2 pts)</b>	<b>Adequate (3 pts)</b>
<b>Title</b>	Missing	Included	-----	-----
<b>Description</b>	Missing	Missing several main ideas	Missing one idea	Includes all main ideas
<b>Prediction</b>	Missing	Included but unclear	Included and clear	-----
<b>Procedure</b>	Missing	Missing several steps or inaccurate steps	Missing one step or partially inaccurate	Includes all steps and is accurate (set up, heights, lengths, and increments)
<b>Data</b>	Missing	Fewer than 10 measurements	Includes all 10 measurements	-----
<b>Results</b>	Missing	Inaccurate power model, missing or poor explanation of the coefficient of determination	Accurate power model & poor explanation of the coefficient of determination	Accurate power model & accurate explanation of the coefficient of determination
<b>Conclusion</b>	Missing	Describes the inverse relationship but does not address limitations of the model	Describes the inverse relationship but inaccurately addresses limitations of the model or vice versa	Accurately describes both the inverse relationship and limitations of the model

➤ Activity 4: Graphs of Rational Functions – Part Two

Allow students to form groups of 3. Write the rational function  $f(x) = \frac{x^2 - 9}{x + 2}$  on the board. Each group will graph the function without the use of a graphing calculator. Students will state the asymptotes, domain, range, and will provide at least one point for each branch of the hyperbola.

*Solutions:*

*Vertical asymptote:*  $x = -2$

*Oblique asymptote:*  $y = x - 2$

*Domain:*  $(-\infty, -2) \cup (-2, \infty)$

*Range:*  $(-\infty, \infty)$

*Points:* Vary

**Advanced Math – Functions and Statistics**  
**Unit 5: Power Functions and Radical Functions**

**Time Frame:** Approximately 2 weeks



**Unit Description**

This unit will continue the study of radical functions begun in Algebra II. Relationships to their power function parents will be explored. Each type of function will be analyzed based on its algebraic and graphic characteristics and properties. The unit will conclude with models of real-life phenomena for both types of functions.

**Student Understandings**

Students will understand the relationship between power functions and radical functions. They will utilize this knowledge to simplify expressions, solve equations, and graph functions. Students will also create mathematical models using power and radical functions. By analyzing graphical and algebraic models, students will understand that power and radical graphs have certain shapes. Students will consider both local and global behavior of power and radical functions and how these concepts affect the fit of the model.

**Guiding Questions**

1. Can students recognize and graph power functions with fractional exponents?
2. Can students state the domain and range of power functions with fractional exponents?
3. Can students describe the end behavior of power functions with fractional exponents?
4. Can students identify local and global extrema of power functions with fractional exponents?
5. Can students model real-life data using power functions with fractional exponents?
6. Can students determine how well their power models fit given or collected data?
7. Can students extrapolate and interpolate values using power models with fractional exponents?
8. Can students state limitations of their power models with fractional exponents?
9. Can students recognize and graph radical functions?
10. Can students state the domain and range of radical functions?
11. Can students describe the end behavior of radical functions?
12. Can students identify local and global extrema of radical functions?
13. Can students model real-life data using radical functions?

**Unit 5 Grade-Level Expectations (GLEs)**

<b>GLE #</b>	<b>GLE Text and Benchmarks</b>
<b>Algebra</b>	
2.	Evaluate and perform basic operations on expressions containing rational exponents (N-2-H)
4.	Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)
6.	Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)
7.	Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)
9.	Solve quadratic equations by factoring, completing the square, using the quadratic formula, and graphing (A-4-H)
10.	Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)
<b>Data Analysis, Probability, and Discrete Math</b>	
20.	Interpret and explain, with the use of technology, the regression coefficient and the correlation coefficient for a set of data (D-2-H)
22.	Explain the limitations of predictions based on organized sample sets of data (D-7-H)
<b>Patterns, Relations, and Functions</b>	
24.	Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)
25.	Apply the concept of a function and function notation to represent and evaluate functions (P-1-H) (P-5-H)
28.	Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)
29.	Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)

**Sample Activities****Activity 1: Power Functions – Fractional Exponents (GLEs: 4, 6, 7)**

Materials List: graphing calculators, Power Functions – Fractional Exponents BLM (one per student), pencils

This activity will utilize a modified *word grid* ([view literacy strategy descriptions](#)) to help students further their understanding of power functions. A modified *word grid* will be used since the labels for the rows are given, and each cell requires a mathematical response instead of just a check or plus sign.

Power functions with fractional exponents of the form  $1/p$  are parent functions for radical functions. The algebraic notation is  $f(x) = kx^{1/p}$ . For this activity, let  $k = 1$ . Make sure that students understand that the one-half power is equivalent to the square root function, the one-third power equivalent to the cube root function, etc.

- Hand out graphing calculators and the Power Functions – Fractional Exponents BLM.
- Allow students to form small groups of 2-3.
- Ask the students to start the *word grid* with  $p = 2$  and continue to  $p = 5$ .
- Students should make predictions about the shape of each graph before graphing each function using the calculator.
- Allow the groups to fill out the *word grid* on their own.
- Give the students time to compare answers with other groups.
- Discuss the results as a class or see Sample Assessments for grading guidelines.
- Ask the class to predict the graph and characteristics for  $f(x) = x^{1/6}$  and  $f(x) = x^{1/7}$ .
- Be sure to discuss generalizations based on whether the value of  $p$  is even or odd. Examine symmetry characteristics for the even and odd value of  $p$ . Tie the graphical symmetry to numerical characteristics. For example,  $f(-x) = -f(x)$  for odd functions.
- Once the teacher feels that the students have mastered the graphs and characteristics included in this activity, ask the students to explore parameter changes for  $k$ . For example, what happens to the graph and characteristics when  $k = 2, k = 3, k = 100, k = 1/2, k = 1/3, k = -1, k = -2$ , etc.?

Students should keep the Power Functions - Fractional Exponents word grids in their notebooks for future reference. Students should be given time to study the word grids before the unit exam. Keep blank Power Functions – Fractional Exponents *word grids* on hand for students that wish to complete it again on their own as a study guide.

Extension: Ask students to explore the following functions as well as their graphs and characteristics.  $f(x) = x^{2/3}$ ;  $f(x) = x^{3/4}$ ;  $f(x) = x^{2/5}$ ;  $f(x) = x^{3/5}$ ; etc.

### Activity 2: Equivalences (GLE: 2, 6)

Materials List: graphing calculators, paper, pencils

In this activity, students will simplify radical expressions by converting them to their fractional power equivalences. If  $a$  is a real number and  $m$  and  $n$  are integers with no common factors with  $n \geq 2$ , then  $a^{m/n} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$ .

- Hand out graphing calculators.
- Allow students to form small groups of 2-3.
- Write the following expression on the board:  $4^{3/2}$ .
- Allow students to plug the expression into the calculator and share their results.
- Ask students to determine how to arrive at the answer without the use of a calculator.

- Let volunteers share their methods with the class. If necessary, remind students that the numerator of the exponent represents the power and the denominator represents the root.
- Let volunteers write the radical equivalents on the board. *Solution:*  $\sqrt{4^3} = \sqrt{64} = 8$  or  $(\sqrt{4})^3 = 2^3 = 8$ . Ask students which equivalent form was easiest to use. *Answers will vary, but generally speaking, it is easier to simplify the radical first in order to keep the numbers small enough to work without a calculator.*
- Write the following expression on the board:  $(-27)^{2/3}$ . This expression will highlight the efficiency of one equivalence form over the other.
- Allow students to plug the expression into the calculators and share their results.
- Ask students to determine how to arrive at the answer without the use of a calculator.
- Let volunteers share their methods with the class. If necessary, remind students that the numerator of the exponent represents the power and the denominator represents the root.
- Let volunteers write the radical equivalents on the board. *Solution:*  $\sqrt[3]{(-27)^2} = \sqrt[3]{729} = 9$  or  $(\sqrt[3]{-27})^2 = (-3)^2 = 9$ .
- Discuss the efficiency of each equivalent form.
- Tell the students to put the covers on the calculators.
- Write the following expressions on the board for each group to solve without the use of a calculator.
  1.  $16^{3/4}$
  2.  $(-8)^{5/3}$
  3.  $9^{-1/2}$
  4.  $(125/27)^{-2/3}$
- Let students check their answers with calculators once they have finished all four problems. *Solutions:* 1. 8    2. -32    3. 1/3    4. 9/25
- Discuss the results. Provide additional sample problems if necessary.

**Extension:** For students that finish early, let them work backwards. Give them expressions in radical form. Students should write the equivalent radical forms and then simplify each

expression. Examples include:  $7^{1/3} \cdot 7^{4/3}$ ,  $6^{1/2} \cdot 6^{2/3}$ , and  $\frac{3^{1/4}}{3^{5/4}}$ . *Solutions:*  $7\sqrt[3]{49}$ ,  $6\sqrt[6]{6}$ ,  $1/3$

### Activity 3: Radical Graphs (GLEs: 4, 6, 7, 25, 28)

Materials List: graphing calculators, paper, pencils, graph paper (optional)

In this activity, students will use their knowledge of translations, dilations, and reflections to graph radical equations. Students will identify the domain and range of each function as well as the  $x$ - and  $y$ -intercepts.

- Hand out graphing calculators.
- Allow students to form groups of 3-4.
- Ask each group to identify the parent function, translations, and dilation of  $f(x) = 2\sqrt{x-1} + 3$ .

- Let volunteers share their results with the class. *Parent:  $f(x) = \sqrt{x}$  ; Translations: right one unit and up 3 units ; Dilation: vertical stretch of factor 2*
- Tell students to graph  $f(x) = 2\sqrt{x-1} + 3$  by hand.
- Let students check their graphs using graphing calculators.
- Ask each group to identify the domain, range, real zero, and y-intercept.
- Let volunteers share their results with the class. *Domain  $[-1, \infty)$  ; Range  $[-3, \infty)$  ; Real zero (x-intercept) = none ; y-intercept = none*
- Ask each group to identify the parent function, translations, dilation, and reflection of  $f(x) = -\sqrt[3]{\frac{1}{2}x} - 1$ .
- Let volunteers share their results with the class. *Parent:  $f(x) = \sqrt[3]{x}$  ; Translation: up 1 unit ; Dilation: horizontal stretch of factor 2 ; Reflection about the x-axis*
- Tell students to graph  $f(x) = -\sqrt[3]{\frac{1}{2}x} - 1$  by hand.
- Let students check their graphs using graphing calculators.
- Ask each group to identify the domain, range, real zero, and y-intercept.
- Let volunteers share their results with the class. *Domain:  $(-\infty, \infty)$  ; Range:  $(-\infty, \infty)$  ; Real zero: -2 ; y-intercept: -1*
- Let group members follow the same process for the following functions:
  1.  $f(x) = -\sqrt{3x+6}$  *Parent:  $f(x) = \sqrt{x}$  ; Translation: left 2 units ; Dilation: horizontal compression of factor 3 ; Reflection about the x-axis ; Domain:  $[-2, \infty)$  ; Range:  $(-\infty, 0]$  ; Real zero: -2 ; y-intercept: 0*
  2.  $f(x) = 5\sqrt{x-1} - 4$  *Parent:  $f(x) = \sqrt{x}$  ; Translation: right one unit & down 4 units ; Dilation: vertical stretch of factor 5 ; Domain:  $[1, \infty)$  ; Range:  $[-4, \infty)$  ; Real zero:  $41/25$  ; y-intercept: none*
  3.  $f(x) = 1/2 \sqrt[3]{x-3} + 1$  *Parent:  $f(x) = \sqrt[3]{x}$  ; Translation: right 3 units & up 1 unit ; Dilation: vertical compression of factor 2 ; Domain:  $(-\infty, \infty)$  ; Range:  $(-\infty, \infty)$  ; Real zero:  $23/8$  ; y-intercept:  $\approx -1.8845$*
  4.  $f(x) = \sqrt[3]{-2x+4}$  *Parent:  $f(x) = \sqrt[3]{x}$  ; Translation: right 2 units ; Dilation: horizontal compression of factor 2 ; Reflection about the y-axis ; Domain:  $(-\infty, \infty)$  ; Range:  $(-\infty, \infty)$  ; Real zero:  $41/25$  ; y-intercept: none*

**Activity 4: Solving Radical Equations (GLEs: 6, 9, 10)**

Materials List: graphing calculators, Solving Radical Equations BLM (one per student), paper, pencils

Students will build on their knowledge of radical equations from Algebra II. The focus of this activity will be on radical equations involving extraneous roots. Students will use modified *story chains* ([view literacy strategy descriptions](#)) to solve each radical equation. At this point in the unit, students are not ready to create their own application situations. Thus, equations are provided at the beginning of the activity. Each group member will work one step of the equation before passing the problem on to the next group member. Each group member works one step until the equation has been solved and checked.

- Hand out graphing calculators.
- Allow students to form groups of 3.
- Write the equation  $x = \sqrt{6 - x}$  on the board.
- Give each group about 5-10 minutes to solve the equation.
- Remind students to check their solutions using graphing calculators.
- Let volunteers share their methods and results with the class. *Solution:  $x = 2$*
- Ask the students to explain why  $x = -3$  is not a solution to the equation. *Solution:  $x = -3$  is an extraneous root because the equation was squared to make it easier to solve. By squaring both sides, a quadratic equation was created. Quadratic equations have two roots. To determine whether or not an extraneous root exists, students should always check their solutions by substituting the values of  $x$  into the original radical equation.*
- Write the equation  $\sqrt{x^2 + 3x} = 2x$  on the board.
- Give each group 5 minutes to solve the equation.
- Again, remind students to check their solutions using graphing calculators.
- Let students share their methods and results with the class. *Solution:  $x = 0$  and  $x = 1$ . This time there is so extraneous root. Both values make the left side of the original radical equation equal the right side. This equation illustrates the need to check both solutions, not just one of them.*
- Hand out the Solving Radical Equations BLM.
- Explain that one student will work the first step of the equation. The rest of the group will either agree or disagree with the first student's work. If they all agree, the next student performs a second step. If someone in the group disagrees, a discussion ensues until either a consensus is reached or the first step is changed. Students continue to pass around the Solving Radical Equations BLM until each equation has been solved and the solutions have been checked.
- Let students know that they can perform more than one step per rectangle if needed.
- Give each group about 20 minutes to complete the Solving Radical Equations BLM.
- Walk around and ask leading questions when necessary.
- See Sample Assessments, Activity-Based, for grading guidelines.
- Let groups that finish early search the Internet for real-life applications of radicals. Students can use these application situations as models for creating their own story chains.
- If time permits, let them share their results with the class.

### Activity 5: Radical Models (GLEs: 25, 29)

Materials List: graphing calculators, paper, pencils

Students will use radical models to solve real-life problems.

- Hand out graphing calculators.
- Allow students to form groups of 3-4.
- Write the formula  $s = 4.62\sqrt[3]{n}$  on the board.
- Tell students that this is the formula for the crew's speed (in m/s) of a rowboat,  $s$ , based on the number of rowers,  $n$ .
- Ask each group to find the expected speed of a rowboat with a crew of 8 rowers.
- Let volunteers share their results with the class. *Solution:  $\approx 5.82$  m/s*
- Ask each group to find the expected number of rowers for a rowboat traveling about 5 m/s. Ask students if the answer will be greater than or less than 8 rowers. *Less than*
- Let volunteers share their results with the class. *Solution: 2 rowers*
- Write the formula  $v = \sqrt{2gh}$  on the board.
- Tell students that this formula describes the velocity of a roller coaster based on gravity and the height of each drop.
- Ask students to find the expected velocity of a roller coaster at the bottom of a 30-foot drop. Do not give students the constant of gravity. Let them figure it out as a group.
- Let volunteers share their results with the class. *Solution:  $\approx 43.8$  ft/s*
- Ask each group to predict the height of a drop if the velocity at the bottom of the drop is 50 m/s.
- Let volunteers share their results with the class. *Solution:  $\approx 127.6$  m*
- Write the formula  $D = \sqrt[3]{216T^2}$  on the board.
- Tell students that the diameter of a major storm (in miles) can be estimated based on the duration of the storm (in hours).
- Ask each group to estimate the diameter of a storm that lasts for 8 hours.
- Let volunteers share their results with the class. *Solution: 24 miles*
- Ask each group to estimate the duration of a major storm with a diameter of 5 miles.
- Let volunteers share their results with the class. *Solution:  $\approx .76$  hour or  $\approx 46$  minutes*

Extension: Let group members search the internet for additional radical models of real-life phenomena.

**Activity 6: Radical Experiment – Pendulum Length vs Period of Swing (GLEs: 20, 22, 24, 25, 29)**

Materials List: graphing calculators, CBLs or EA 100s, motion detectors, fishing weights (or the equivalent), string, yardsticks, Pendulum Experiment BLM (one per group), pencils

Galileo discovered that the period of a pendulum depends on gravity and the length of the pendulum. He found that the period and the length of a pendulum are related by a fractional power/radical function. Students will discover the relationship as a result of this experiment.

NOTE: It is always recommended that the teacher perform any experiment before he or she tries it with the class.

- Allow students to form groups of 4-5 based on the availability of equipment.
- Hand out the Pendulum Experiment BLM.
- Let group members read the setup section and collect their materials.
- Write the following table on the board:

Length (in)						
Period (sec)						

- Walk around and assist group members.
- Tell students to record their data on the board.
- When all groups have recorded their data, tell students to enter the data into L1 and L2 in the graphing calculators.
- Students should find a power/radical function to model the data.
- Students should state the domain and range of the models.
- Students should use their models to interpolate and extrapolate values.
- Students should state limitations for the models.
- See Sample Assessments for possible grading guidelines.

Extension: Students that finish the experiment early should try to construct a pendulum with a period of exactly 3 seconds.

**Activity 7: Mathematical Masterminds (GLEs: 2, 4, 6, 7, 9, 10, 20, 22, 24, 25, 28, 29)**

Materials List: graphing calculators, paper, pencils, professor attire – optional (ties, pocket protectors, crowns, tiaras, thinking caps, etc.)

Mathematical masterminds is a spin-off of the literacy strategy *professor know-it-all* ([view literacy strategy descriptions](#)). This strategy is used when coverage of new content for the unit has been completed.

- Allow students to form groups of 3-4.
- Ask students to review all material from this unit (Activities 1-6).

- Give each group time to create 3-5 questions to ask the mathematical masterminds.
- Ask one group to come to the front of the room. Tell the class that the students standing before them are mathematical masterminds. Their job is to answer questions posed by the class.
- Allow a student to ask a question.
- Tell the mathematical masterminds to huddle together and come up with an answer. One of the masterminds should share the answer with the class.
- The class either agrees with the solution or challenges the mathematical masterminds to defend and/or revise their answer.
- Once the class and the teacher are satisfied with the answer, another student asks a question and the process repeats itself.
- Assign a time and/or question limit so that several groups get a chance to come up to the front of the room.
- If the class is leaving out important content questions, the teacher should lead them toward the questions that need to be asked or he/she should ask a leading question

### Sample Assessments

#### General Assessments

- Each student will create a third entry for his or her *learning log* ([view literacy strategy descriptions](#)). Remember, students will be adding entries to the *learning logs* throughout the school year. The fourth entry will have the title, “Extraneous Roots”. This assessment piece can be formal or informal in nature. Informally, students would be assigned a grade for writing the entry. This will let you know what students do and do not understand about graphing radical functions. Formally, students can be graded based on the inclusion of main topics like converting a radical equation into a quadratic equation, solving the quadratic equation (factoring, completing the square, or quadratic formula), domain restrictions, and checking solutions by hand as well as with technology (graphing calculator).
- Each student will continue to add terms from this unit to the glossary started in Unit 1. This glossary will be included in a student portfolio that will be graded at the end of each semester.
- Each student will complete a “spiral” comprised of 5 -10 teacher-made problems. These problems should center on the topics covered in Units 1, 2, 3, and 4. Again, at least half of the problems should be in multiple choice form. This will help prepare students for the unit test and the ACT/SAT exams. “Spirals” should be assigned every 2-3 weeks to ensure that students understand and retain important concepts and procedures.

### Activity-Specific Assessments

➤ Activity 1: Power Functions – Fractional Exponents

Since students are familiar with *word grids* from previous units, let them turn in the Power Functions - Fractional Exponents BLM for a grade. For example, students could be awarded two points for each correct entry, 1 point for each incorrect entry, and 0 points for no entry. This would make the assignment worth 28 points. You could leave the point total at 28 or make it worth 25 and allow for 3 bonus points. Use the Power Functions – Fractional Exponents with Answers BLM to award point values.

➤ Activity 4: Solving Radical Equations

Allow students to form groups of 3. Write the following radical equations on the board:

$$1) x = \sqrt{15 - 2x} \qquad 2) \sqrt{5x + 6} = x + 2 \qquad 3) \sqrt{3x - 5} - \sqrt{x + 7} = 2 .$$

Group members should work the problems on their own and then compare answers. The point total of this small quiz is left to the discretion of the teacher.

➤ Activity 6: Radical Experiment – Pendulum Length vs Period of Swing

Each student or group could create and turn in a lab report for the experiment. Possible headings for the lab report include: title, description, prediction (of what will happen to the period as the length increases/decreases), procedure, data, results, and conclusion. While rubrics and point totals are always left to the teacher's discretion, a sample rubric is provided below.

Points	Missing (0 pts)	Poor (1 pt)	Lacking (2 pts)	Adequate (3 pts)
<b>Title</b>	Missing	Included	-----	-----
<b>Description</b>	Missing	Missing two main components	Missing one component	Includes all 3 components (period, length, & gravity)
<b>Prediction</b>	Missing	Included but unclear	Included and clear	-----

<b>Procedure</b>	Missing	Missing several steps or inaccurate steps	Missing one step or partially inaccurate	Includes all steps and is accurate
<b>Data</b>	Missing	Inaccurate measurements	Accurate measurements	-----
<b>Results</b>	Missing	Inaccurate power model, missing or poor explanation of the coefficient of determination	Accurate power model & poor explanation of the coefficient of determination	Accurate power model & accurate explanation of the coefficient of determination
<b>Conclusion</b>	Missing	Describes the relationship but does not address limitations of the model	Describes the relationship but inaccurately addresses limitations of the model or vice versa	Accurately describes both the relationship and limitations of the model

**Advanced Math – Functions and Statistics**  
**Unit 6: Exponential Functions and Logarithmic Functions**

**Time Frame:** Approximately 3.5 weeks



**Unit Description**

This unit focuses on graphing functions, simplifying expressions, solving equations, and modeling real-life situations and data using exponential and logarithmic functions. The unit expands on the properties of exponents and logarithms covered in prior math courses and provides a review of essential mathematical skills needed in this course and in future courses.

**Student Understandings**

Students recognize, evaluate, and graph exponential and logarithmic functions. The laws of exponents and logarithms are reviewed and then utilized to simplify expressions and solve equations. Students will use both exponential and logarithmic functions to model and solve real-life problems as well as recognize what kinds of data grow or decay exponentially.

**Guiding Questions**

1. Can students graph simple exponential functions?
2. Do students understand the meaning of logarithms?
3. Can students graph simple logarithmic functions?
4. Can students graph translations, dilations, and reflections of exponential functions?
5. Can students graph translations, dilations, and reflections of logarithmic functions?
3. Can students use the properties of exponents and logarithms to simplify expressions and solve equations?
4. Do students understand the difference between discrete and continuous growth?
6. Can students model growth and decay situations using exponential functions?
7. Can students determine if a set of data can be modeled with an exponential function?
8. Can students solve real-life applications of logarithms?
9. Can students determine if a set of data is best modeled by a linear function, a power function, or an exponential function?
10. Can students linearize a set of exponential data?

**Unit 6 Grade-Level Expectations (GLEs)**

<b>GLE #</b>	<b>GLE Text and Benchmarks</b>
<b>Number and Number Relations</b>	
3	Describe the relationship between exponential and logarithmic equations (N-2-H)
<b>Algebra</b>	
4.	Translate and show the relationships among non-linear graphs, related tables of values, and algebraic symbolic representations (A-1-H)
6.	Analyze functions based on zeros, asymptotes, and local and global characteristics of the function (A-3-H)
7.	Explain, using technology, how the graph of a function is affected by change of degree, coefficient, and constants in polynomial, rational, radical, exponential, and logarithmic functions (A-3-H)
8.	Categorize non-linear graphs and their equations as quadratic, cubic, exponential, logarithmic, step function, rational, trigonometric, or absolute value (A-3-H) (P-5-H)
9.	Solve quadratic equations by factoring, completing the square, using the quadratic formula, and graphing (A-4-H)
10.	Model and solve problems involving quadratic, polynomial, exponential, logarithmic, step function, rational, and absolute value equations using technology (A-4-H)
<b>Data Analysis, Probability, and Discrete Math</b>	
19.	Correlate/match data sets or graphs and their representations and classify them as exponential, logarithmic, or polynomial functions (D-2-H)
20.	Interpret and explain, with the use of technology, the regression coefficient and the correlation coefficient for a set of data (D-2-H)
<b>Patterns, Relations, and Functions</b>	
24.	Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)
27.	Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology (P-3-H)
28.	Represent and solve problems involving the translation of functions in the coordinate plane (P-4-H)
29.	Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)

## Sample Activities

### Activity 1: Graphs of Exponential Functions (GLEs: 4, 7, 8)

Materials List: graphing calculators, Graphs of Exponential Functions BLM, pencils, paper, grid paper (optional)

In this activity, students will investigate the parental form of the exponential function family;  $f(x) = b^x$ , when  $b > 1$  and  $0 < b < 1$ . Students will use technology to discover the shape and properties of the exponential family of functions.

A modified *opinionnaire* ([view literacy strategy descriptions](#)) will be utilized to promote meaningful understandings of the exponential family of functions by activating prior knowledge and building interest to extend learning.

- Hand out graphing calculators and the Graphs of Exponential Functions BLM
- Ask students to read each statement. Tell students to place a “√” (check) in the column marked My Opinion if they agree with the statement or place an “X” if they disagree with the statement. Both the reading of the statements and the marking of opinions should be done independently.
- Allow students to compare opinions with a partner. If opinions differ, students should defend their positions.
- Ask students to graph the following exponential functions on the same screen of the graphing calculator:  $y = 1.5^x$ ,  $y = 2^x$ ,  $y = 3^x$ .
- Tell students to sketch the graphs on their own paper. Note: A single grid should be used for all three graphs.
- Students should write down similarities and differences for the three graphs. T-charts may make it easier for students to compare the rates of change.
- Remind students to examine properties as well as the shapes.
- Ask students to predict the properties and shape of  $y = 4^x$  and sketch this function on the same grid as the other three exponential functions.
- Let students compare their predictions and sketches to the graph generated by the graphing calculator. If necessary, students should adjust their answers.
- Ask students to graph the following exponential functions on the same screen of the graphing calculator:  $y = 0.3^x$ ,  $y = 0.5^x$ ,  $y = 0.8^x$ .
- Tell students to sketch the graphs on their own paper. Again, a single grid should be used for all three graphs.
- Students should write down similarities and differences for the three graphs. Again, T-charts may make it easier for students to compare the rates of change.
- Remind students to examine properties as well as the shapes.
- Ask students to predict the properties and shape of  $y = 0.4^x$  and sketch this function on the same grid as the other three exponential functions.
- Let students compare their predictions and sketches to the graph generated by the graphing calculator. If necessary, students should adjust their answers.

- Give students time to fill in the column labeled “Calculator Findings” on the Graphs of Exponential Functions BLM.
- Allow students to compare and adjust answers before discussing the “Calculator Findings” as a class.
- Next, students should complete the “Lessons Learned” column of the Graphs of Exponential Functions BLM by answering true or false. If false, explain why.
- Allow students to compare and adjust answers before discussing the “Lessons Learned” as a class.
- For statement #1, ask students marking “True” to stand on one side of the classroom and those marking “False” to stand on the other side.
- Let students on each side debate the statement. Students changing their minds should be allowed to move to the other side of the classroom.
- If possible, continue the debate until all students are on one side of the classroom.
- Repeat the process for statements 2-6.
- Ask students to predict the graph and properties for  $f(x) = e^x$ .  
*e = Euler’s number  $\approx 2.718$  (irrational) ; since the base is greater than 1, the graph will be increasing; the domain is still  $(-\infty, \infty)$ ; the range is still  $(0, \infty)$ ; the asymptote is the x-axis ( $y = 0$ ); the function is concave up; the y-intercept is 1*

Students should keep the Graphs of Exponential Functions *opinionnaire* in their notebooks for future reference. Students should be given time to study the *opinionnaire* before the unit exam. Keep blank *opinionnaires* on hand for students that wish to complete it again on their own as a study guide.

### Activity 2: Graphs of Logarithmic Functions (GLEs: 3, 4, 6, 7, 27)

Materials List: graphing calculators, Graphs of Logarithmic Functions BLM, pencils

In this activity, students will investigate the parental form of the logarithmic function family;  $f(x) = \log_b x$ , when  $b > 1$  and  $0 < b < 1$ . Students will use technology to discover the shape and properties of the logarithmic family of functions.

A modified *word grid* ([view literacy strategy descriptions](#)) will be utilized to elicit meaningful student participation in learning important terms and concepts. Since the cells will be filled in with graphs, equations, intervals, points, and words, instead of just checks and plus signs, a modified *word grid* is needed. To accomplish this task, the class will select logarithmic functions with  $b > 1$  to place in the first row. Next, students will discuss and decide which function properties to list in the first column. If students leave important items out, use leading questions to guide them in the right direction.

- Hand out graphing calculators and the Graphs of Logarithmic Functions BLM.
- Ask the class to name logarithmic functions with  $b > 1$ . List their responses on the board. If they do not mention the common log or natural log, ask the students if they remember the names of any special logarithmic functions. Write both the natural log and the common log functions on the board.

- Ask the class to select two regular log functions and two special log functions to include in the first row of the Graphs of Logarithmic Functions BLM. Give the students time to write them down on the BLM.
- Next, ask the class to name properties of the logarithmic functions that they would like to explore. Refer to the Graphs of Logarithmic Functions with Answers BLM to make sure that important properties are not left out.
- Allow students to form small groups of 2-3.
- Group members will work together to complete each cell of the Graphs of Logarithmic Functions BLM.
- After about 15-20 minutes, allow groups to compare answers.
- Discuss the results as a class. Make sure that students understand the inverse relationship between exponential functions and log functions. Illustrate this point by graphing  $f(x) = e^x$  and  $f(x) = \ln x$  on the same coordinate plane and by comparing T-charts. Students should discover that the domain of one function is the range of the other and vice versa. Students should also note that the graphs are reflections about the line  $y = x$ .

Students should keep the Graphs of Logarithmic Functions *word grids* in their notebooks for future reference. Students should be given time to study the *word grid* before the unit exam. Keep blank *word grids* on hand for students that wish to complete it again on their own as a study guide.

### **Activity 3: Translations, Dilations, and Reflections of Exponential & Logarithmic Functions (GLEs: 3, 4, 7, 8, 27, 28)**

Materials List: graphing calculators, Translations, Dilations, and Reflections of Exponential Functions BLM, Translations, Dilations, and Reflections of Log Functions BLM, pencils

Students will use knowledge gained from previous activities and units to sketch translations, dilations, and reflections of exponential and logarithmic functions.

- Hand out graphing calculators and the Translation, Dilations, and Reflections of Exponential Functions BLM.
- Allow students to form groups of 3-4.
- Ask students to complete the first two rows of the Translations, Dilations, and Reflections of Exponential Functions BLM without using calculators.
- Let students check their answers using graphing calculators and make revisions if necessary.
- Give each group about 15-20 minutes to complete the rest of the Translations, Dilations, and Reflections of Exponential Functions BLM.
- Let the groups compare answers before discussing the results as a class.
- Hand out the Translations, Dilations, and Reflections of Log Functions BLM.
- Again, ask students to complete the first two rows without using calculators.
- Ask the students which log functions can be checked with calculators.  
*Solution: the common log and the natural log*

- Give each group about 20-25 minutes to complete the Translations, Dilations, and Reflections of Log Functions BLM.
- Tell students that this BLM will be turned in for a grade. See Sample Assessments for scoring guidelines.

#### Activity 4: Solving Exponential Equations (GLEs: 6, 10)

Materials List: paper, pencils, calculators

This activity reviews and expands upon knowledge gained in Algebra II. Students will solve exponential equations with and without a calculator. Whenever possible, students will use the same base method. When it is impossible to rewrite the problem using common bases, students will utilize the inverse function (common log or natural log) and the power property to rewrite the equation.

- Allow students to form groups of 3-4.
- Write the following equation on the board:  $5^{3x} = 5^{7x-2}$ .
- Give each group a couple of minutes to come up with an answer.
- Let volunteers share their results with the class. If the answers are different, ask the class how they can check to see which answer is correct. *Solution:  $x = \frac{1}{2}$*
- Remind students that they used the same base method to find the answer.
- Write the following exponential equation on the board:  $3^{2x} = 27^{x-1}$ .
- Give each group a couple of minutes to generate an answer.
- Let volunteers share their results with the class. If the answers are different, make students defend their methods and solutions. *Solution:  $x = 3$*
- If students were able to solve the first two equations with ease, write the following equations on the board for them to solve as groups:

$$1. 27^{1-x} = \left(\frac{1}{9}\right)^{2-x} \quad 2. 5^{x^2} = 125 \quad 3. 10^{x-1} = 0.0001 \quad 4. \frac{1}{8^{x-2}} = 4^{5-3x}$$

*Solutions: 1)  $x = 7/5$  2)  $x = \pm\sqrt{3}$  3)  $x = -3$  4)  $x = 4/3$*

- Hand out calculators.
- Briefly review the power property of logs and the change-of-base formula.

Power Property:  $\log a^b = b \bullet \log a$     Change-of-base formula:  $\log_b a = \frac{\log a}{\log b}$

ex.  $\log 5^3 = 3 \log 5$

ex.  $\log_3 7 = \frac{\log 7}{\log 3}$

- Remind students that the change-of-base formula is important since the calculator is only programmed to evaluate common logs and natural logs.
- Write the following exponential equation on the board:  $2^x = 19$ .

- Ask students if the same base method can be used to solve this equation. *No, 19 is not a power of 2.*
- Ask students to approximate the answer. *x will be between 4 and 5 since  $2^4 = 16$  and  $2^5 = 32$ . Furthermore, x will be closer to 4 since 16 is closer to 19.*
- Ask students to state the inverse of an exponential function with a base of 2.  
*Solution: log with a base of 2*
- On the board, write  $\log_2$  on both sides of the equation.  $\log_2 2^x = \log_2 19$
- Ask a volunteer to apply the power property of logs to simplify the left side of the equation.  $x \log_2 2 = \log_2 19$
- Ask students to evaluate  $\log_2 2$ . *1*
- Therefore, write  $x = \log_2 19$  on the board.
- Ask a volunteer to apply the change-of-base formula to rewrite the right side of the equation.  $x = \frac{\log 19}{\log 2}$
- Ask students to approximate the answer to the nearest thousandth.  $x \approx 4.248$
- If students struggle with the change-of-base formula, work the problem in the following manner. Take either the common log or natural log of both sides of the equation.  
 $\log 2^x = \log 19$  OR  $\ln 2^x = \ln 19$
- Apply the power property of logs.  $x \log 2 = \log 19$  OR  $x \ln 2 = \ln 19$
- Solve for x.  $x = \frac{\log 19}{\log 2} \approx 4.248$  OR  $x = \frac{\ln 19}{\ln 2} \approx 4.248$
- Write the following equation on the board for each group to solve using its method of choice:  $e^{x-2} = 76$ .
- Ask volunteers to share their methods and results with the class.  $x \approx 6.331$
- Write the following problems on the board for each group to solve. Let students solve the equations using their methods of choice.

1.  $3^x = 75$                       2.  $5^{x-1} = 999$                       3.  $10(\frac{1}{2})^{3x} = 50$                       4.  $2(10)^{-x+3} = 1002$

5.  $60e^{2x} + 1 = 3037$                       6.  $5^{2-x} = 2^{x+1}$                       7.  $(0.4)^{x+2} = 1.8^{2x-1}$

- Let volunteers share their results with the class.

*Solutions:* 1)  $x \approx 3.930$     2)  $x \approx 5.291$     3)  $x \approx -0.774$     4)  $x \approx .300$   
5)  $x \approx 1.962$     6)  $x \approx 1.097$     7)  $x \approx -1.157$

### Activity 5: Solving Logarithmic Equations (GLEs: 3, 6, 9, 10)

Materials List: Solving Logarithmic Equations BLM, paper, pencils

Logarithmic functions are a part of the college algebra curriculum. Most algebra II students do not master solving log equations the first time around. This activity is designed to provide additional opportunities for students to master this high-level skill.

- Hand out the Solving Logarithmic Equations BLM
- Allow students to form groups of 3-4.
- Tell students that the steps necessary to solve the first equation are given in the wrong order. It is their job to put them in the correct order.
- Let volunteers share their results with the class.
- Each group will use the steps in #1 to try and solve the next two *log* equations.
- Let volunteers share their results with the class. Be sure to explain that  $x = -3$  is not a solution since the domain of the *log* function is  $(0, \infty)$ .
- Briefly review the following *log* properties:

$$\log(a \cdot b) = \log a + \log b$$

$$\log(a/b) = \log a - \log b$$

$$b \log a = \log a^b$$

$$\text{If } \log a = \log b, \text{ then } a = b.$$

- Ask each group to apply the log properties to correctly order the steps in equation #4.
- Discuss the results.
- Give each group about 15-20 minutes to solve the rest of the equations on the Solving Logarithmic Equations BLM.
- Let volunteers share their results with the class.

### Activity 6: Exponential Growth & Decay (GLEs 10, 24)

Materials List: calculators, paper, pencils, Exponential Growth & Decay BLM, Money Investments BLM

There are two kinds of exponential growth & decay problems; discrete and continuous. If the growth is discrete, it can be modeled by the formula  $A = A_0b^t$ ; where  $A$  represents the final amount,  $A_0$  represents the initial amount,  $b$  is the growth or decay factor, and  $t$  is the elapsed time.

If the growth is continuous, then  $b$  is equal to  $e^k$ . If  $b > 1$  (exponential growth) then  $k > 0$ . If  $b < 1$  (exponential decay) then  $k < 0$ . The growth or decay formula can be written as  $A = A_0e^{kt}$ .

This is also known as the Law of Uninhibited Growth/Decay since  $A$  is said to be growing or decaying at a continuous rate of  $k$ .

#### Part One

- Hand out calculators.
- Allow students to work in groups of 3-4.
- Write the following situation on the board: Ice is added to a glass of water. If not stirred, the water at the bottom cools according to the formula  $F(t) = 72(0.94)^t$ , where  $F$  is the temperature in degrees Fahrenheit and  $t$  is the number of minutes since the ice was added.
- Ask students to determine the real-life meaning of 0.94. *Since  $0.94 < 1$ , it represents the decay factor.*
- Ask students to determine the initial temperature of the water.  $72^\circ F$
- Ask students to determine the temperature of the water 5 minutes after the ice was added.  $F(5) \approx 52.8^\circ F$

- Ask students to determine how long it will take for the water to cool to 60°F.  
 $t \approx 2.9 \text{ min}$
- Write the following situation on the board: After a week of rain and high humidity, the mosquito population obeyed the law of uninhibited growth. One thousand mosquitoes were found in the trap initially and 1700 were found on day 2.
- Ask students to find the growth constant.  $k \approx 0.2653$
- Ask students to write an algebraic model representing the number of mosquitoes over time.  $M(t) \approx 1000e^{0.2653t}$ ; where  $t$  is the time measured in days
- Ask students to find the mosquito population on day 4.  $M(t) \approx 2890 \text{ mosquitoes}$
- Ask students to determine how long will it take for the population to reach 15,000.  
 $t \approx 10.2 \text{ days}$
- Write the following situation on the board: The half-life of radium is 1690 years.
- Ask each group to find the value of  $k$  (decay factor) for radium.
- Let volunteers share their results with the class.  $\frac{1}{2} = e^{1690k}$ ;  $k \approx -0.0004$
- Ask each group to use the decay formula to determine how much of a 10 g sample of radium will remain after 100 years.
- Let volunteers share their results with the class.  $A \approx 9.608 \text{ g}$
- Hand out the Exponential Growth & Decay BLM.
- Each group will create a *story chain* ([view literacy strategy descriptions](#)) modeled after the examples covered in class. A *story chain* involves a small group of students writing a story problem. The first student starts the story. The next student adds a sentence and passes it to the next student to do the same. If a group member disagrees with any of the previous sentences, the group discusses the work that has already been done. They either agree to revise the problem or move on as it is written.
- After each *story chain* is written, at least three questions should be generated.
- To assist students in the creation of a *story chain*, write the first line of a *story chain* on the board. (Ex. Initially, there are two rabbits.)
- Let a volunteer come to the board to write the next line. (Ex. After 13 days, there are 8 rabbits.)
- Let a volunteer come to the board to write the first question. (Ex. Find a model to represent the total number of rabbits after  $t$  days.)
- Let a volunteer come to the board to write the second question. (Ex. How many rabbits will there be in three weeks?)
- Let a volunteer come to the board to write a third question. (Ex. How long will it take for there to be 50 rabbits?)
- Note: Each group will not answer its own questions. Answers will be generated by a different group.
- Give each group ample time to create its own *story chain*.
- Each group will swap papers with another group.
- Give each group 10-15 minutes to answer the questions.
- If there are any concerns about a question, group members should consult the writers of the problem for clarification. If necessary, a neutral party may have to review the student-generated problems and questions.
- Papers should be handed back to the original authors. They will check the answers and provide feedback to the group that generated the answers. If necessary, give students additional time to adjust their answers.

- Note: Providing feedback is essential because the authors will be graded on how well the other group answered the questions.

### Part Two

- Hand out calculators.
- Allow students to form groups of 3-4.
- Write the investment formulas:  $A = P\left(1 + \frac{r}{n}\right)^{nt}$  and  $A = Pe^{rt}$  on the board. Remind students that the first formula is for money that is being compounded a discrete number of times annually, and the second formula is for investments that are compounded continuously.
- Ask students to state what each variable represents.  $A = \text{final amount}$  ;  $P = \text{principal amount}$  ;  $r = \text{rate}$  ;  $n = \text{number of times compounded per year}$  ;  $t = \text{time in years}$
- Write the situation on the board: Jose plans to invest \$5,000 in a five-year Certificate of Deposit that pays an interest rate of 4.5%, compounded annually.
- Ask each group to write an algebraic model for this situation.
- Discuss the models.  $A = 5000(1 + .045)^t$
- Ask each group to determine how much money Jose will have after the 5-year period?
- Let volunteers share their results with the class.  $A \approx \$6,230.91$
- Ask each group to determine if Jose would have been better off investing his money at an interest rate of 4.25%, compounded quarterly for 5 years.
- Let volunteers share their results with the class. *Yes, he would earn \$6,253.75*
- Ask each group to find an interest rate that would give Jose \$7500 at the end of 5 years if compounded semiannually.
- Let volunteers share their results with the class.  $r \approx 8.28\%$
- Ask each group how long it would take Jose to double his money if invested at 3%, compounded continuously.
- Let volunteers share their results with the class.  $t \approx 23.1 \text{ years}$
- Each group will create a second *story chain* modeled after the investment examples covered in part two of the activity.
- After each *story chain* is written, at least three questions should be generated.
- Each group will swap papers with the same group that answered the questions from the first *story chain*.
- Give each group 5-10 minutes to answer the questions.
- If there are any concerns about a question, group members should consult the writers of the problem for clarification. If necessary, a neutral party may have to review the student-generated problems and questions.
- Papers should be handed back to the original authors. They will check the answers and provide feedback to the group that generated the answers. If necessary, give students additional time to adjust their answers.
- Again, providing feedback is essential because the authors will be graded on how well the other group answered the questions.
- After feedback has been provided and corrections made, papers should be returned to the authors. Each group will select one of the two *story chains* to be graded. Note: All group members should agree with the selection.
- Each group will turn in its selected *story chain*.
- See Sample Assessments for scoring guidelines.

### Activity 7: Applications of Logarithmic Functions (GLEs: 10, 24)

Materials List: computer lab, paper, pencils, Loudness of Sound BLM, Magnitude of Earthquakes BLM, calculators

While logarithms are often used to solve real-life problems modeled by exponential functions, there are common real-life phenomena that are modeled directly with logarithmic functions. These phenomena are magnitudes of earthquakes, intensity of sound, and loudness of sound. Both sound and earthquakes are measured using logarithmic scales.

This activity will incorporate the *student questions for purposeful learning* strategy ([view literacy strategy descriptions](#)). This strategy helps students ask questions that are important to them before reading and learning. By doing so, students heighten anticipation and engage in more purposeful exploration of the topic as they search for answers to their questions. To start this strategy, a statement about the applications of logarithms will be shared with the students. This *SQPL* statement may be true or false.

- Allow students to form small groups of 2-3.
- Hand out the Loudness of Sound BLM.
- Ask a volunteer to read the *SQPL* statement to the class.
- Each group should generate 3 questions based on the *SQPL* statement.
- Ask each group to share their questions with the class. A volunteer or two will write the questions on the board as they are shared with the rest of the class.
- As questions are repeated, they should be marked with an asterisk.
- If any pertinent questions are left out, share your own questions to complete the list.
- Students will research the topic of Loudness of Sound using the Internet. Students will try to answer their own questions as well as those posed by their classmates. Students should record their answers so that they can later share them with the class.
- After about 20 minutes, bring the class back together.
- Refer to the list on the board and ask students to share their answers with the class.
- Give students time to record answers to questions that they did not find on their own.
- Next, ask each group to answer the questions on the Loudness of Sound BLM.
- Let volunteers share their results with the class.
- Repeat the process using the Magnitude of Earthquakes BLM.

### Activity 8: Linearizing Exponential Data (GLEs: 3, 4, 8, 19, 20, 24, 29)

Materials List: graphing calculators, paper pencils, Linearizing Data BLM

In Unit 2, students learned how to find a line of best fit for a set of data and to interpret the correlation coefficient for that data. In this activity, students will look at sets of non-linear data and determine which function will give the best model. Graphing calculators give the correlation coefficient for the linear, exponential, logarithmic, and power functions. They also calculate exponential regressions by running linear regressions on  $x$  and  $\log y$ .

Exponential Function

$$y = ab^x, b \neq 1$$

$$\log y = \log(ab^x)$$

$$\log y = \log a + x \log b$$

Substituting  $A$  for  $\log a$  and  $B$  for  $\log b$  gives the linear form  $\log y = A + Bx$ .

The graph is of the form  $(x, \log y)$ .

Part One

- Hand out graphing calculators.
- Allow students to form groups of 3-4.
- Ask each group to graph  $y_1 = 150(1.5)^x$  using a viewing window of  $x_{\min} = 0$ ,  $x_{\max} = 5$ ,  $y_{\min} = 0$  and  $y_{\max} = 1000$ .
- Ask each group to graph  $y_2 = \log y_1$  using a viewing window of  $x_{\min} = 0$ ,  $x_{\max} = 5$ ,  $y_{\min} = 0$  and  $y_{\max} = \log 1000$ .
- Ask volunteers to describe the graph of  $y_2$ . *It is linear.*
- Ask each group to find the coordinates of 2 points on the line, keeping all of the digits that the calculator provides. *Students can find the two points by using the trace feature on the graphing calculator or by substituting in values for  $x$  and recording the  $y$  values.*
- Ask each group to find the slope between the two points.
- Ask volunteers to share their results with the class.  $m \approx .1760912591$
- Write an equation for  $y_2$  in  $a + bx$  form. Do not round off the values for  $a$  and  $b$ .
- Ask volunteers to share their equations.  $y = .1760912591x + 2.176091259$
- Ask each group to find a relationship between the slope of  $y_2$  and the base of  $y_1$ . *The slope of  $y_2$  is  $\log(1.5)$ .*
- Ask each group to find a relationship between the  $y$ -intercept of  $y_2$  and the coefficient of  $y_1$ .
- Ask volunteers to share their results with the class. *The  $y$ -intercept is  $\log(150)$ .*
- Ask each group why linearization of data is important. *Non-linear data is often difficult to characterize. Non-linear data can be represented by several different functions like polynomial, power, exponential, logistic, and trigonometric. The manner which data is linearized will determine which curve is the best fit. In this case, students should come to the conclusion that the points  $(x, y)$  lie on an exponential curve,  $y = ab^x$  if and only if the points  $(x, \log y)$  lie on a line.*

Part Two

- Hand out graphing calculators.
- Allow students to work in groups of 3-4.
- Write the following situation on the board:

In 2000 an automobile had a retail price of \$24,300. A local dealership used the following guide to determine the approximate value of the car over the next 5 year period.

Year	2000	2001	2002	2003	2004	2005
Value (\$)	24,300	21,700	21,200	19,640	18,465	17,350

- Ask each group to enter the data on the graphing calculator and draw the scatterplot using an appropriate window. Enter the year 2000 as 0, 2001 as 1, and so on.

- Ask each group to determine which of the following functions best fits the data: linear, exponential, or power. Students will have to defend their answers.
- Let each group share their methods and results with the class.
  - linear regression:  $r \approx -.9853$*
  - exponential regression:  $r \approx -.9905$*
  - power regression:  $r \approx -.7912$  (the 0 must be entered as .00001 because  $\log 0$  does not exist)*
  - Since the exponential regression had the best correlation coefficient, the model  $y = 23851.32(0.9379)^x$  is the best fit for the data.*
- Tell students to graph the exponential regression equation on the scatterplot using the graphing calculators. Ask students if the model is a good fit.
- Ask each group to defend this “good” fit by linearizing the data.
- Let volunteers share their results with the class.  $y \approx -0.0278x + 4.3775$
- Hand out the Linearizing Exponential Data BLM.
- Give each group about 25-30 minutes to complete the BLM.
- Tell students that this BLM will be turned in for a grade.
- See Sample Assessments for scoring guidelines.

## Sample Assessments

### General Assessments

- Each student will create a fifth entry for his or her *learning log* ([view literacy strategy descriptions](#)). Remember, students will be adding entries to the *learning logs* throughout the school year. The fifth entry will have the title, “Exponential and Logarithmic Functions – Similarities and Differences.” This assessment piece can be formal or informal in nature. Informally, students would be assigned a grade for writing the entry. This will let you know what students do and do not understand about exponential and logarithmic functions. Formally, students can be graded based on the inclusion of main topics like domain, range,  $x$ -intercepts,  $y$ -intercepts, asymptotes, rates of change, concavity, increasing/decreasing intervals, and properties.
- Each student will continue to add terms from this unit to the glossary started in Unit 1. This glossary will be included in a student portfolio that will be graded at the end of each semester.
- Each student will complete a “spiral” comprised of 5 -10 teacher-made problems. These problems should center on the topics covered in Units 1, 2, 3, 4, and 5. Again, at least half of the problems should be in multiple choice form. This will help prepare students for the unit test and the ACT/SAT exams. “Spirals” should be assigned every 2-3 weeks to ensure that students understand and retain important concepts and procedures.

## Activity-Specific Assessments

### ➤ Activity 3: Translations, Dilations, and Reflections of Exponential & Logarithmic Functions

There are 36 cells in Activity 3's BLM. Award students one point for each correct response. Use the Translations, Dilations, and Reflections of Exponential & Logarithmic Functions With Answers BLM to check student responses.

### ➤ Activity 6: Exponential Growth & Decay

Award the group that authored the *story chain* a score of 5 points for creating an appropriate real-life situation based on either exponential growth or decay. Award 3-4 points for creating an exponential growth or decay problem with minor flaws. Award 1-2 points for the creation of a exponential growth or decay problem with major flaws. Do not award any points if the situation created is not exponential in nature.

Next, examine the questions. Award the group that authored the questions a score of 5 points for creating appropriate questions. Reduce the number of points awarded if the questions are incomplete or do not fit the situation.

Lastly, award each group that authored the questions, points for the answers. Each correct answer is worth 5 points. Reduce the number of points awarded if the answers are incorrect or incomplete. This makes the overall total 35 points.

### ➤ Activity 8: Linearizing Data

Award each group 10 points for identifying the function of best fit. The values of the correlation coefficient must be compared and used to justify the selection of the function of best fit. Award 8 points if the correlation coefficients are correct and the function of best fit is correctly identified, but the explanation is missing. Award 5 points if the function of best fit is given, but the correlation coefficients and the explanation are missing. Award between 0 and 4 points if the function of best fit is not correctly identified, but some work is shown.

Award each group 5 points for generating the correct model. Award 3 points if the base is correct but the coefficient is incorrect, or vice versa. Award 1 to 2 points if the model is incorrect. Award 0 points if a model is not given.

Award 5 points if the linearization is correct and the work is shown. Award three points if the work is not shown but the linearization is correct. Award 1 to 2 points if the work is shown but is incorrect. Award 0 points for failing to write the linearization model.

## Advanced Math – Functions and Statistics

### Unit 7: Univariate Statistics

**Time Frame:** Approximately 3.5 weeks



#### Unit Description

This unit begins with the development of a statistical vocabulary. The difference between descriptive and inferential statistics is introduced. Next, students discover and utilize methods of collecting, organizing, and analyzing univariate (single-variable) data. Advantages and disadvantages of various data displays are examined as are measures of center and spread. The unit ends with an exploration of distribution shapes and applications of the normal distribution curve.

#### Student Understandings

Students will display data using different types of graphical presentations. Students will become proficient at calculating and interpreting descriptive statistics by hand for small data sets. Students will use technology to calculate these statistics for larger and more complex sets of data. They will also explain the difference between inferential and descriptive statistics.

#### Guiding Questions

1. Can students explain the meaning of the terms inferential statistics and descriptive statistics and discuss their uses?
2. Can students organize and display data in a variety of ways including: line plots, bar graphs, circle graphs, stem-leaf plots, and box plots?
3. Can students organize and display data using frequency tables, histograms, relative frequency tables, and relative frequency histograms?
4. Can students calculate and interpret measures of central tendency?
5. Can students calculate and interpret measures of spread?
6. Can students describe the shape of the data in terms of skewness, and can they express the relationship of the skew to the mean and median of the data set?
7. Can students draw normal distribution curves given the mean and standard deviation?
8. Can students solve application problems using the normal distribution curve?
9. Can students convert raw scores to z-scores?
10. Can students solve application problems using the standard normal distribution curve and/or a table of z values?

**Unit 7 Grade-Level Expectations (GLEs)**

GLE #	GLE Text and Benchmarks
<b>Data Analysis, Probability, and Discrete Math</b>	
17.	Discuss the differences between samples and populations (D-1-H)
21.	Describe and interpret displays of normal and non-normal distributions (D-6-H)
22.	Explain the limitations of predictions based on organized sample sets of data (D-7-H)

**Sample Activities****Activity 1: Descriptive and Inferential Statistics Vocabulary (GLEs: 17, 22)**

Materials List: computer lab, Vocabulary Cards BLM (teacher only), index cards, paper, pencils

In this activity, students will explore the vocabulary of statistics. Mathematicians and organizations like the National Council for Teachers of Mathematics (NCTM) state that there is a critical need for students to develop a solid math vocabulary. Too many students lack the vocabulary needed to understand the concepts they are studying, to explain their work, and to solve real-life application problems. Therefore, this activity will incorporate the creation and use of *vocabulary cards* ([view literacy strategy descriptions](#)). Each *vocabulary card* will have the term to be defined written on one side. A definition, preferably in a student's own words, and a real-life example will be written on the other side of the card.

- Take students to a computer lab. If possible, each student should have his or her own computer. If not, let students work in groups of 2-3.
- Note: If access to a computer lab is not possible, make a class set of copies of either Internet statistics pages or pages from a statistics textbook.
- Distribute 13 index cards to each student.
- Write the word statistics on the board.
- Ask students what the word statistics means to them.
- Write student responses on the board.
- Discuss the responses. If appropriate, combine responses to strengthen the definition. Eliminate any that do not fit the definition of statistics.
  - Statistics is the science of collecting, organizing, and analyzing data.*
- Tell students that they will create a *vocabulary card* for the term statistics by writing the word statistics on one side of an index card.
- Tell students to select one of the remaining definitions and write it on the other side of the *vocabulary card*. On the same side, ask students to give an example of how statistics are used in the real world.
- Discuss the real-life examples.
  - Example: Statistics are used to determine car insurance rates.*

- Write the following vocabulary terms on the board: individual, variable, quantitative variable, qualitative variable, population, sample, nominal data, ordinal data, interval data, ratio data, descriptive statistics, and inferential statistics.
- Tell students to find definitions and examples using the Internet.
- Students should make a vocabulary card for each term.
- Give students time to compare definitions and examples.
- Discuss the results as a class. It is important that each student has appropriate definitions and examples before moving on to the next activity.
- Tell students to punch a hole in each card in order to secure the cards in their binders.
- Students should be reminded to study the *vocabulary cards* throughout the unit. If possible, allow partners to hold each other accountable by giving them time in class to quiz each other using the *vocabulary cards*.
- See the Vocabulary Cards BLM for sample definitions and real-life examples.

### Activity 2: Collecting and Organizing Univariate Data (GLEs: 17, 21)

Materials List: Collecting and Organizing Univariate Data BLM, Data Displays: Advantages and Disadvantages BLM, calculators, paper, pencils

In this activity, students will collect and organize univariate data. Students will display data in line plots, circle graphs, bar graphs, stem-leaf plots, and box plots. For most students, this will be a review from previous math courses. Students will also demonstrate an understanding of the different types of data displays by completing a modified *word grid* ([view literacy strategy descriptions](#)). This word grid requires students to write information in the boxes instead of putting simple checks, plusses, or minuses.

- Ask students to walk around the classroom and record the shoe size of each student in the class.
- Ask students why this is an example of univariate data.
- Let volunteers share their ideas with the class.  
*There is only one variable – shoe size.*
- Ask students if the data set is from a sample or a population.
- Let volunteers share their ideas with the class.  
*The data set is from a population since data was collected from every person in the classroom.*
- Ask students to brainstorm ways to organize the data.
- List student responses on the board.  
*Make sure that the following graphs are included: line plots, bar graphs, circle graphs, stem-leaf plots, and box-whiskers plots.*
- Let volunteers come to the board to teach the class how to draw the graphs mentioned above.
- Discuss the results.
- If students need additional practice with any of the above graphs, give them a sample of quiz grades to organize.
- Let volunteers come to the board and draw each of the above graphs.

- Discuss the results.
- Allow students to form small groups of 2-3.
- Hand out the Collecting and Organizing Univariate Data BLM.
- Each student will complete and turn in the Collecting and Organizing Univariate Data BLM for a grade. See Sample Assessments for scoring guidelines.
- Hand out the Data Displays: Advantages and Disadvantages BLM.
- Small groups will work together to complete the Data Displays: Advantages and Disadvantages BLM.
- Let volunteers share their answers with the class. Discussions should continue until a consensus is reached.
- Check the Data Displays: Advantages and Disadvantages with Answers BLM to make sure that all major points have been discussed and recorded.

### Activity 3: Frequency Tables and Histograms (GLE: 21)

Materials List: Frequency Tables and Histograms BLM, Math Test Grades BLM, graphing calculators, paper, pencils

In this activity, students will enhance their knowledge of histograms and frequency tables. They will create frequency tables, relative frequency tables, histograms, and relative frequency histograms for given data sets.

- Hand out graphing calculators and the Frequency Tables and Histograms BLM.
- Ask students if there are advantages to displaying the data in the BLM in a frequency table.
- Let volunteers share their ideas with the class.  
*A frequency table would provide a condensed display for the data set.*
- Ask students what must be determined before constructing a frequency table.
- Let volunteers share their ideas with the class.  
*The number of classes must be determined. Most frequency tables have between 5 and 15 classes.*
- Ask students why having fewer than 5 classes or more than 15 classes could result in a poor display of data.  
*Data can be hidden when fewer than 5 classes are used. Using more than 15 classes can lose the effectiveness of grouping the data in the first place.*
- Tell students that there is not a right way or wrong way to determine the appropriate number of classes. Experimentation with different values is usually the best way to determine which number of classes will produce the most informative display.
- Ask students what they think would be an appropriate number of classes.
- Let volunteers share their responses and reasoning with the class.
- Since this is the first example, tell the class to use 10 classes to create a frequency table. This will make it easier to answer questions and keep everyone on task.
- Write the class width formula on the board.

$$\text{Class width} = \frac{\text{largest data value} - \text{smallest data value}}{\text{number of classes}}$$

Note: The class width or bin width is rounded to the next largest whole number.

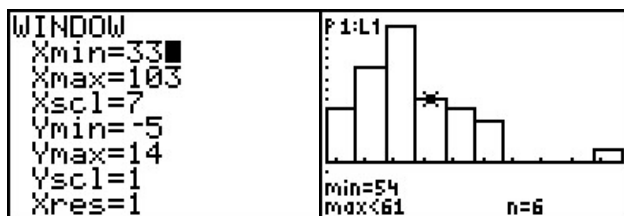
- Ask students to find the class width using 10 classes.
- Let volunteers share their results with the class.

$$\text{Class width} = \frac{101 - 33}{10} = 6.8 \text{ which rounds to } 7$$

- Tell students to fill in the lower and upper limits for each class on the BLM.
- Let students compare answers with each other. Walk around to make sure that everyone's limits are correct.
- Now that the class width is known, ask students to find and record the number of birds in each interval.
- Discuss the results as a class.
- Next, ask students to find the class midpoint for each interval. If necessary, remind students how to find the midpoint of two values.

$$\text{Class midpoint} = \frac{\text{upper class limit} + \text{lower class limit}}{2}$$

- Let volunteers share their midpoints with the class.
- Ask students how histograms differ from bar graphs.  
*Histograms must be vertical, the bars must touch each other, and the data must be continuous.*
- Ask students to create a histogram on the back of the Frequency Tables and Histograms BLM. Note: There are different ways of choosing class boundaries. One way is to use the lower class limits as class boundaries. When a data point is on the class boundary, count it with the class to the right. Most graphing calculators use this method. To label the histogram, use either the lower class limits or the midpoints.
- For this example, ask students to label the histogram using the midpoint of each class.
- Let students compare histograms before asking volunteers to come to the board to draw one bar each until the histogram is complete.
- Show students how to use a calculator or computer to draw a histogram based on bin or class widths that have been computed.
- Tell students to enter the bird lengths into List 1. Tell students to press STAT PLOT (2<sup>nd</sup> Y=) and highlight the histogram. Tell students to press GRAPH. If there are problems with the histogram, have the students set the window to the values below.



Note: Make sure that students understand that histograms produced by graphing calculators and computers may differ based on how each has been programmed to select the number of classes and the class width.

- Next, ask students how a relative frequency table differs from a frequency table.
- Let volunteers share their ideas with the class.

*A relative frequency table uses percentages. This is advantageous when the actual frequencies involve large numbers which can cause scaling problems.*

- Ask students how to find the relative frequency for each class.
- Let volunteers share their ideas with the class.

$$\text{Relative frequency} = \frac{f}{n} \quad (f \text{ is the class frequency and } n \text{ is the size of the sample})$$

- Ask students to find and record the relative frequency for each class.
- Ask students to find the sum of the relative frequencies. Ask students to interpret the sum.
- Let students compare answers.
- Discuss the results as a class.
 

*The relative frequencies should add up to 1 since 100% of the data should be included. When rounding is involved, the sum may be slightly greater than or less than 1.*
- Ask students to graph the relative frequency histogram on the back of the Frequency Tables and Histograms BLMs.
- Make sure that the students label the y-axis values as relative frequencies (or percents) instead of actual frequencies (numbers).
- Ask students to compare the shapes of the frequency histogram and the relative frequency histogram.
- Let volunteers share their insights with the class.
 

*The shapes are the same.*
- Tell students that relative frequency tables and histograms can also be used to determine percentile ranks.
- Ask students to define percentile rank.
- Let volunteers share their definitions with the class.
 

*A percentile rank is the percent of scores that are below a given value.*
- Ask students to use their relative frequency tables or histograms to find the percentile rank for a duck with an average length of 54 cm.
- Let volunteers share their methods and results with the class.
 

*63<sup>rd</sup> percentile (.30 + .21 + .12)*
- Tell students that they will see relative frequency distributions again in Unit 9 when probability distributions and inferential statistics are explored.
- Hand out the Math Test Grades BLM.
- Let students work in small groups of 2-3.
- Students will turn in the Math Test Grades BLM for a grade. See Sample Assessments for scoring guidelines.

#### **Activity 4: Descriptive Statistics (GLEs: 17, 22)**

Materials List: Tropical Cyclones BLM, graphing calculators, paper, pencils

In this activity, students will practice calculating descriptive statistics for a data set. The descriptive statistics will be divided into two categories: measures of central tendency and measures of spread.

- Ask students to name and describe the three types of tropical cyclones.  
*Tropical depressions: some rotational movement and winds  $\leq 38$  mph*  
*Tropical storms: circular shape and sustained winds  $\geq 39$  mph but  $\leq 73$  mph*  
*Hurricanes: sustained winds  $\geq 74$  mph*
- Ask students to state the naming requirements for tropical cyclones.  
*When a tropical cyclone has sustained winds of at least 39 mph, it receives a name. The names are issued in alphabetical order and alternate between female and male names.*
- Hand out the Tropical Cyclones BLM.
- Ask students what must be true if Grace was the last tropical cyclone named in 1991.  
*There had to be storms with names beginning with A, B, C, D, E, and F before Grace. Thus, there were a total of 7 named tropical cyclones in 1991.*
- Ask students to define descriptive statistics and give a real-life application of how they can be used to describe tropical cyclones statistics.  
*Descriptive statistics describe data using numbers. Common measures include: mean, median, mode, range, interquartile range, variance, and standard deviation. The strength of tropical cyclones can be described using descriptive statistics.*
- Ask students to name measures of central tendency. If necessary, remind students that they have been finding measures of central tendency since elementary school.  
*Mean: average Median: middle value Mode: most frequent value(s)*  
*Each of these measures represents the data set using a single value.*
- Ask students to find the measures of central tendency for the number of hurricanes in the sample given in the Tropical Cyclones BLM.
- Discuss the results and introduce the summation symbol in the mean formula.  

$$\text{Mean: } \bar{x} = \frac{\sum x}{n} = \frac{83}{13} \approx 6.38 \quad \text{Median: } 7 \quad \text{Mode: Bimodal } 3 \text{ \& } 4$$
*Make sure that students understand that a data set can have one mode or two modes. If there are three or more data values that share the largest frequency count, the data set does not have a mode.*
- Ask students to name the measures of spread.  
*Range: difference between the largest value and the smallest value*  
*Interquartile range: difference between the third and the first quartiles*  
*Variance: average of the squared differences from the mean*  
*Standard Deviation: measures the spread of the data about the mean using the square root of the variance*
- Asks students to find the measures of spread for the number of hurricanes in the sample. Note: Since the data set is large, students should not try to find the interquartile range by hand. They should wait until the data is entered into the graphing calculator.
- Students will be able to find the range on their own, but the formulas for calculating the variance and standard deviation of a sample will have to be introduced and discussed.  

$$\text{Variance: } s^2 = \frac{\sum (x - \bar{x})^2}{n - 1} \quad \text{Standard Deviation: } s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}}$$
- A deviation is the difference of a data value from the mean ( $x - \bar{x}$ ). Ask students why the deviations are squared.  
*Some data values will be greater than the mean. Others will be lower than the*

mean. The sum of these positive and negative differences from the mean will cancel each other out and result in a sum of zero unless they are squared.

- Discuss the results.

$$\text{Range: } 11 - 3 = 8$$

$$\text{Variance: } s^2 = \frac{\sum (x - \bar{x})^2}{n - 1} \approx \frac{105.08}{12} \approx 8.76$$

$$\text{Standard Deviation: } s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}} \approx \sqrt{8.76} \approx 2.96$$

- Ask students why the standard deviation is more commonly used than the variance. *The variance describes the spread of the data in squared units. The standard deviation takes the square root so that the measure of spread is in the same units as the original data values.*
- Tell students to enter the number of hurricanes for each year in List 1 of the graphing calculator.
- Tell students to press CALC and then 1VAR to find the univariate statistics.
- Tell students to check the results the class got by hand with the ones generated by technology. Note: On most calculators the sample standard deviation is labeled  $S$ ,  $s_x$ , or  $\sigma_{n-1}$ .
- Students should use the univariate statistics generated by the calculator to determine the interquartile range.  $Q_3 - Q_1 = 9 - 3.5 = 5.5$  (*IQR: Fifty percent of the data lies between 3.5 and 9.*)
- Ask students if any of the data values seem quite different from the rest. Such a data value is called an outlier. Since opinions will vary about which data values might be outliers, a mathematical formula should be used. Any data value greater than  $Q_3 + 1.5$  times the interquartile range is considered an outlier. Any data value less than  $Q_1 - 1.5$  times the interquartile range is also considered an outlier.
- Ask students to determine if there are any outliers for the hurricane data set.
- Let volunteers share their results with the class.  $Q_3 + 1.5 IQR = 9 + 1.5(5.5) = 17.25$   
 $Q_1 - 1.5 IQR = -4.75$  *Since there are no data values greater than 17.25 or less than -4.75, there are no outliers.*
- Ask students what the descriptive statistics reveal about the number of hurricanes in the sample.  
*Since the mean was about 6 and the standard deviation was about 3, there should be between 3 to 9 hurricanes for each year in the sample.*
- Be careful. Descriptive statistics are not used to make predictions. Descriptive statistics describe what has already occurred.
- Ask students to determine the total number of cyclones for each year by counting the letters up to the last named Cyclone. For example, in 1993, the last storm was Harvey (H). The other storms started with A, B, C, D, E, F, and G, for a total of 8 storms.
- Discuss the results.

Year	Last Named Tropical Cyclone	Total Number of Cyclones
1991	Grace	7
1992	Frances	6
1993	Harvey	8
1994	Gordon	7
1995	Tanya	20
1996	Marco	13
1997	Grace	7
1998	Nicole	14
1999	Lenny	12
2000	Nadine	14
2001	Olga	15
2002	Lili	12
2003	Peter	16

- Ask students to find the measures of central tendency for the total number of cyclones.
- Discuss the results.

$$\text{Mean: } \bar{x} = \frac{\sum x}{n} = \frac{151}{13} \approx 11.62 \quad \text{Median} = 12 \quad \text{Mode: none}$$

- Ask students to find the measures of spread.
- Discuss the results.

$$\text{Range: } \max - \min = 20 - 6 = 14$$

$$\text{Interquartile range: } Q_3 - Q_1 = 14.5 - 7 = 7.5$$

$$\text{Variance: } s^2 = \frac{\sum (x - \bar{x})^2}{n - 1} = \frac{223.08}{12} \approx 18.59$$

$$\text{Standard Deviation: } s = \sqrt{\frac{\sum (x - \bar{x})^2}{n - 1}} = \sqrt{18.59} \approx 4.31$$

- Ask students if there are any outliers.
- Let volunteers share their results with the class.  $14.5 + 1.5(7.5) = 25.75$   
 $7 - 1.5(7.5) = -4.25$  Since no data values are greater than 25.75 or less than -4.25, there are no outliers.
- Ask students what the descriptive statistics reveal about the total number of tropical cyclones for the data set.  
*Since the mean is about 12 and the standard deviation is about 4, there should be between 8 and 16 tropical cyclones for each year in the sample.*
- Again, be careful. Descriptive statistics are not used to make predictions. Descriptive statistics describe what has already occurred.

**Extension:** If time permits, show students how to calculate the variance and standard deviation using the sum of squares formula.

$$SS_x = \sum x^2 - \frac{(\sum x)^2}{n} \quad s^2 = \frac{SS_x}{n-1} \quad s = \sqrt{\frac{SS_x}{n-1}}$$

### Activity 5: More Descriptive Statistics (GLEs: 17, 22)

Materials List: calculators, paper, pencils

In this activity, students will learn how to compare the spread or variability of two different types of data sets using the coefficient of variation. Students will also use Chebyshev's Theorem to determine intervals of measures based on the number of standard deviations from the mean.

- Write the following descriptive statistics on the board.  
In 2004, the national average on the verbal section of the SAT was 508 with a standard deviation of 112. The national average on the English section of the 2004 ACT was 20.4 with a standard deviation of 5.9.
- Ask students to compare the standard deviations from two different sets.
- Discuss the results.  
*There is no way to compare the averages and standard deviations since the units of the two tests are different as are their populations. Note: The dependence on units is a flaw of the standard deviation.*
- The coefficient of variation resolves the problem of trying to compare two data sets with different units of measure. It expresses the standard deviation as a percentage of the sample or population mean. Therefore, there are no units.

$$CV = \frac{s}{\bar{x}} \cdot 100 \quad (\text{sample}) \quad CV = \frac{\sigma}{\mu} \cdot 100 \quad (\text{population})$$

- Ask students to find the coefficient of variation for the 2004 SAT and the ACT.
- Let volunteers share their results with the class.  
 $CV_{SAT} = \frac{112}{508} \cdot 100 \approx 22.05\% \quad CV_{ACT} = \frac{5.9}{20.4} \cdot 100 \approx 28.92\%$
- Ask students what the percentages reveal about the data sets.
- Let volunteers share their interpretations with the class.  
*The variability of the two tests can now be compared. The 2004 ACT English sub-scores shows more variability when compared to the mean than the verbal scores of the 2004 SAT.*
- Another method for examining measures of spread is Chebyshev's theorem. Write the theorem on the board. Students should make a vocabulary card for this theorem.

Chebyshev's Theorem: For any set of data and for any constant  $k$  greater than 1, the proportion of data that must lie within  $k$  standard deviations on either side of the mean is *at least*  $1 - \frac{1}{k^2}$ .

- Ask students to find the proportion of 2004 SAT data that lies within 2 standard deviations of the mean and then determine an interval of scores for that percentage.

- Let volunteers share their results with the class.

$$1 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4} = 75\% \quad \text{Therefore, at least 75\% of the SAT students would have a verbal sub-score between 284 and 732.}$$

- Repeat the process for the 2004 ACT data.
- Let volunteers share their results with the class.

$$1 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4} = 75\% \quad \text{Therefore, at least 75\% of the ACT students would have an English sub-score between 8.6 and 32.2.}$$

- Write the following data on the board.

A national park ranger found the sample mean number of deer per square kilometer to be 21.3 with a standard deviation of 9.2. The ranger also found the sample mean number of bass in a lake to be 133.4 with a standard deviation of 49.8.

- Ask students to compare the variability of the deer and bass for homework.
- Also, ask students to find the interval of deer and bass that Chebyshev's theorem guarantees to be within 3 standard deviations of the mean.

$$CV_{\text{deer}} = \frac{9.2}{21.3} \cdot 100 \approx 43.2\% \quad CV_{\text{bass}} = \frac{49.8}{133.4} \cdot 100 \approx 37.3\%$$

Thus, there is more variability or spread in the values of the deer data set compared to the mean than in the bass data set.

$$1 - \frac{1}{3^2} = 1 - \frac{1}{9} = \frac{8}{9} = 88.9\% \quad \text{Therefore, at least 88.9\% of the park and lake sample bass. Note: The number of deer and bass cannot be negative.}$$

### Activity 6: Shapes of Distributions (GLE: 21)

Materials List: Distribution Shapes BLM, paper, pencils

Distribution shapes can be classified as symmetric, uniform or rectangular, skewed right, skewed left, and bimodal. In this activity, students will fill in a chart by matching names, definitions, and real-life examples with their appropriate distribution shapes.

- Hand out the Distribution Shapes BLM.
- Allow students to form small groups of 2-3.
- Without any instruction, ask students to complete the chart by matching the names, definitions, and real-life examples with the appropriate distribution shapes.
- Give the groups time to compare and adjust answers.
- Let volunteers share their results with the class.
- Compare student results with the Distribution Shapes with Answers BLM.
- Tell students that the shape of a distribution, in histogram form, may provide information about the relative sizes of the median and mode of the data set. However, the converse does not hold true.

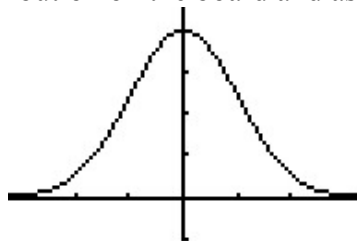
- Ask students which distribution shapes would have an identical mean and median.
- Let volunteers share their ideas and reasoning with the class. Encourage students to illustrate their reasoning on the board.  
*symmetric distribution and uniform distribution*
- Ask students which distribution shape would have a mean  $<$  median.
- Let volunteers share their ideas and reasoning with the class. Encourage students to illustrate their reasoning on the board.  
*skewed left distribution*
- Ask students which distribution shape would have a mean  $>$  median.
- Let volunteers share their ideas and reasoning with the class. Encourage students to illustrate their reasoning on the board.  
*skewed right distribution*
- Students should keep the charts in their notebooks to study throughout the unit. Extra blank copies of the Distribution Shapes BLM should be made available to students for extra practice.
- Tell students to create their own real-life examples for each type of distribution shape for homework.
- Let students compare homework answers.
- Let volunteers share their results with the class.

### Activity 7: The Normal Distribution (GLE: 21)

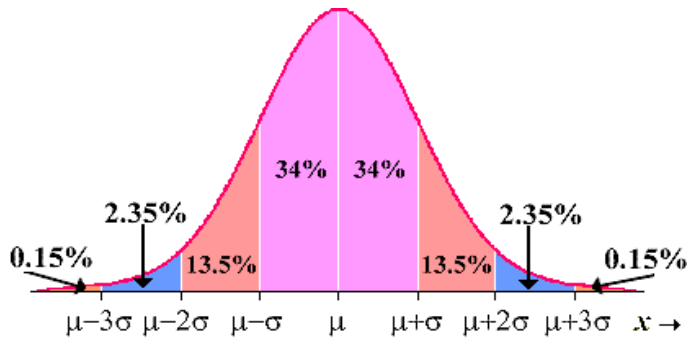
Materials List: Normal Distribution BLM, calculators, paper, pencils

While skewed distributions come in many shapes, there is only one normal shape. Even though some normal distributions appear tall and thin while others appear short and fat, the scale of the x-axis can be adjusted so that they look identical.

- Draw the normal distribution on the board and ask students to name its properties.

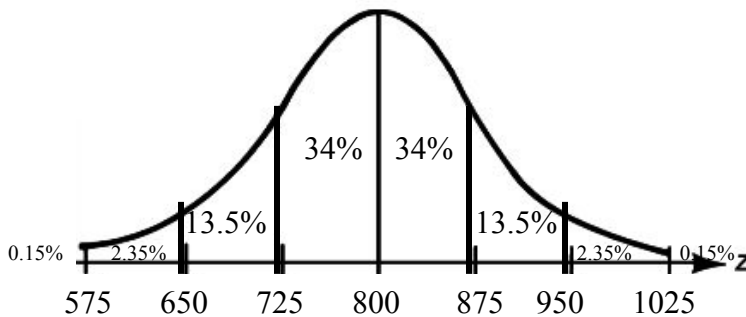


- Let volunteers come to the board and share their properties with the class.  
*It is symmetrical. It is bell-shaped. The highest point lies directly above the mean. The mean divides the area in half. The curve approaches but never crosses the horizontal axis. The mean, median, and mode coincide.*
- Illustrate the remaining properties by labeling the normal distribution.



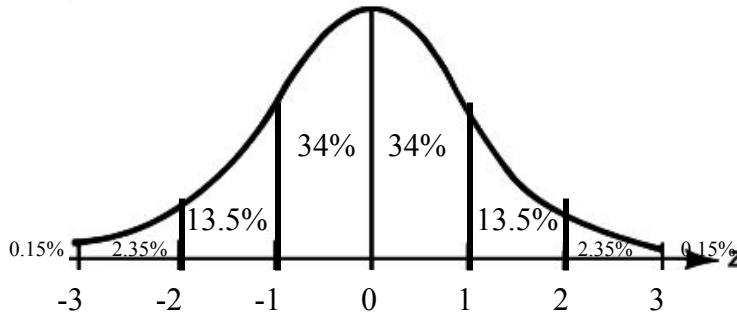
*In normally distributed data about 68% of the values lie within 1 standard deviation of the mean; about 95% of the values lie within 2 standard deviations of the mean; and about 99% of the values lie within 3 standard deviations of the mean. Also, the total area under the curve equals one.*

- Ask students what concavity means and how it relates to the normal distribution.
- Let volunteers come to the board and share their ideas with the class.  
*Concavity refers to the curvature of graph. Concave up indicates a curve that opens upward, and concave down indicates a curve that opens downward. In terms of the normal distribution, the concavity changes at  $\mu \pm \sigma$ .*
- Ask students to draw a normal distribution for a light bulb with a mean life of 800 hours and a standard deviation of 75 hours.
- Let volunteers come to the board to draw the normal distribution one part at a time.



- Ask students to find the probability that this type of light bulb will last between 875 hours and 950 hours.
- Let volunteers come to the board to share their methods and results.  
*13.5%*
- Ask students to find the probability that this type of light bulb will last 800 hours at most.
- Let volunteers come to the board to share their methods and results.  
*50%*
- Ask students the following question. From a sample of 1200 light bulbs, how many can be expected to last between 575 hours and 650 hours?
- Let volunteers come to the board to share their methods and results.  
 *$1200 \cdot 0.235 \approx 28$  light bulbs*

- Ask students to create their own problems based on the normal distribution of this particular type of light bulb.
- Let volunteers come to the front of the classroom to share their problems with the class. Note: Students should know the answers to their own problems before presenting them to the class.
- Draw the standard normal distribution on the board. Ask students how it differs from the normal distribution.



- Let volunteers share their results with the class.  
*σ is 1 and μ is 0.*
- Tell students that the variable along the x-axis is called a z-score. Z-scores are standard values that can be found by using the formula:  $z = \frac{x - \bar{x}}{s}$ .
- Probabilities for values lying on, before, or after the boundaries of each region of a standard normal distribution are found in the same manner as probabilities on a normal distribution.
- To find probabilities for values between boundaries of the standard normal distribution, a Z-table can be used.
- Write or project the following Z-table on the board.

<b>Z</b>	<b>Area</b>	<b>Z</b>	<b>Area</b>	<b>Z</b>	<b>Area</b>
0.0	0.000	1.0	0.341	2.0	0.477
0.1	0.044	1.1	0.364	2.1	0.482
0.2	0.079	1.2	0.385	2.2	0.486
0.3	0.118	1.3	0.403	2.3	0.489
0.4	0.155	1.4	0.419	2.4	0.492
0.5	0.192	1.5	0.433	2.5	0.494
0.6	0.226	1.6	0.445	2.6	0.495
0.7	0.258	1.7	0.455	2.7	0.496
0.8	0.288	1.8	0.464	2.8	0.497
0.9	0.316	1.9	0.471	2.9	0.498

- Ask students to find the probability that a light bulb will last exactly 600 hours.
- Let volunteers share their methods and results with the class.

Since 600 hours lies between boundaries on the normal distribution, a z-score should be found.  $z = \frac{600 - 800}{75} = -2.\bar{6} \approx -2.7$  Reading the Z-table, the probability is  $\approx 49.6\%$ .

- Ask students to find the probability that a light bulb will last at least 900 hours.
- Let volunteers share their methods and results with the class.  
 Since 900 hours lies between boundaries on the normal distribution, a z-score should be found.  $z = \frac{900 - 800}{75} = 1.\bar{3} \approx 1.3$  Reading the Z-table, the probability for lasting exactly 900 hours is  $\approx 40.3\%$ . Therefore, the probability of lasting longer than 900 hours is  $1 - .403 \approx 59.7\%$
- Ask students to create their own problems based on the standard normal distribution of this particular type of light bulb.
- Let volunteers come to the front of the classroom to share their problems with the class. Students should know the answers to their own problems before presenting them to the class.
- Hand out the Normal Distribution BLM.
- Allow students to work in small groups of 2-3.
- Each student will turn in the Normal Distribution BLM for a grade. See Sample Assessments for scoring guidelines.

## Sample Assessments

### General Assessments

- Each student will create a seventh entry for his or her *learning log* ([view literacy strategy descriptions](#)). Remember, students will be adding entries to the *learning logs* throughout the school year. The seventh entry will have the title, “Univariate Data.” This assessment piece can be formal or informal in nature. Informally, students would be assigned a grade for writing the entry. This will let you know what students do and do not understand about univariate statistics. Formally, students can be graded based on the inclusion of data graphs, descriptive statistics (measures of central tendency and spread), and distribution shapes (defined, illustrated, and explained).
- Each student will continue to add terms from this unit to the glossary started in Unit 1. This glossary will be included in a student portfolio that will be graded at the end of each semester.
- Each student will complete a “spiral” comprised of 5 -10 teacher-made problems. These problems should center on the topics covered in Units 1, 2, 3, 4, 5, 6, and 7. Again, at least half of the problems should be in multiple choice form. This will help prepare students for the unit test and the ACT/SAT exams. “Spirals” should be assigned every 2-3 weeks to ensure that students understand and retain important concepts and procedures.

## Activity-Specific Assessments

### ➤ Activity 2: Collecting and Organizing Univariate Data

Each student will turn in his or her own Collecting and Organizing Univariate Data BLM. For #1, assign 3 points for the collection of the sibling data and 2 points for the population identification. For #2, assign one point for each of the correct parts of the box plot (minimum, 1<sup>st</sup> quartile, median, 3<sup>rd</sup> quartile, and maximum) for a total of 5 points. For #3, assign 5 points for an accurate graph. For #4, assign 5 points for an accurate graph. This brings the overall total of the Collecting and Organizing Univariate Data BLM to 20 points.

### ➤ Activity 3: Frequency Tables and Histograms

Each student will turn in his or her own Math Test Grades BLM. Award 1 point for each correct cell of the table for a total of 16 points. Then, award 4 points for the frequency histogram and its labels. This brings the overall total of the Math Test Grades BLM to 20 points.

### ➤ Activity 7: The Normal Distribution

Each student will turn in his or her own Normal Distribution BLM. Count each correct answer as 2 points for a grand total of 20 points.

**Advanced Math – Functions and Statistics**  
**Unit 8: Bivariate Statistics**

**Time Frame:** Approximately 2.5 weeks



**Unit Description**

This unit covers the statistics of two related variables. The topics include displaying and analyzing paired data and formalizing relationships which have been introduced in previous years. In particular, techniques for quantitatively describing the relationship between independent and dependent variables will be presented.

**Student Understandings**

Students will deepen their understanding of what it means for ordered pairs to have a linear relationship, and how to quantify the relationship. They will understand the meaning of positive, negative, and 0 correlation. Students will understand why the method of least squares produces a line that fits a set of related data. They will also understand the meanings of coefficient of determination and correlation coefficient. Students will understand how to make predictions based on a mathematical model and how to determine the usefulness of their predictions.

**Guiding Questions**

1. Can students display bivariate data in an appropriate manner?
2. Can students calculate and interpret residuals?
3. Given a set of ordered pairs with a linear relationship, can students calculate the least squares line?
4. Can students state the real-life meaning of the regression slope and y-intercept?
5. Can students calculate and interpret the coefficient of determination and correlation coefficient?
6. Can students explain the effect of outliers and influential points on the least squares line?
7. Can students state limitations of regression equations?
8. Can students draw and interpret residual plots?
9. Can students transform data to achieve linearity?

**Unit 8 Grade-Level Expectations (GLEs)**

<b>GLE #</b>	<b>GLE Text and Benchmarks</b>
<b>Data Analysis, Probability, and Discrete Math</b>	
19.	Correlate/match data sets or graphs and their representations and classify them as exponential, logarithmic, or polynomial function (D-2-H)
20.	Interpret and explain, with the use of technology, the regression coefficient and the correlation coefficient for a set of data (D-2-H)
22.	Explain the limitations of predictions based on organized sample sets of data (D-7-H)
<b>Patterns, Relations, and Functions</b>	
24.	Model a given set of real-life data with a non-linear function (P-1-H) (P-5-H)
27.	Compare and contrast the properties of families of polynomial, rational, exponential, and logarithmic functions, with and without technology (P-3-H)
29.	Determine the family or families of functions that can be used to represent a given set of real-life data, with and without technology (P-5-H)

**Sample Activities****Activity 1: Bivariate Vocabulary Cards (GLEs: 20, 29)**

Materials List: computer lab, Bivariate Vocabulary Cards BLM (teacher only), index cards, paper, pencils

In this activity, students will explore the vocabulary of bivariate statistics. As in the previous unit, *vocabulary cards* ([view literacy strategy descriptions](#)) will be utilized. Keep in mind that true understanding of words is more than memorizing definitions. Students should be able to use vocabulary words in context and provide real-life examples.

- Take students to a computer lab. If possible, each student should have his or her own computer. If not, let students work in groups of 2-3.
- Note: If access to a computer lab is not possible, make a class set of copies of either Internet statistics pages or pages from a statistics textbook.
- Distribute 13 index cards to each student.
- Write the word bivariate on the board.
- Ask students what the word bivariate means to them.
- Write student responses on the board.
- Discuss the responses.
  - Having two variables*
  - Tell students to write the word bivariate on one side of an index card.
  - Tell students to write the definition on the other side of the index card. On the same side, ask students to give a real life example of a bivariate situation.
  - Discuss the real-life examples.
    - Example: number of calories eaten in a day and a person's weight.*

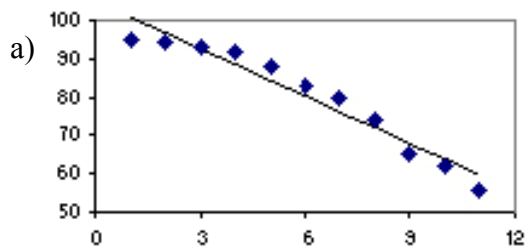
- Write the following vocabulary terms on the board: scatterplot, correlation, correlation coefficient, coefficient of determination, residual, regression line, least squares line, explanatory variable, response variable, extrapolation, interpolation, and causation.
- Tell students to find definitions and examples using the Internet.
- Students should make a *vocabulary card* for each term.
- Give students time to compare definitions and examples.
- Discuss the results as a class. It is important for each student to write appropriate definitions and examples before moving on to the next activity.
- Tell students to punch a hole in each card in order to secure the cards in their binders.
- Students should be given time to quiz each other throughout the unit using the *vocabulary cards*.
- See the Bivariate Vocabulary Cards BLM for sample definitions and real-life examples.

### Activity 2: Scatterplots and Linear Correlations (GLEs: 20, 29)

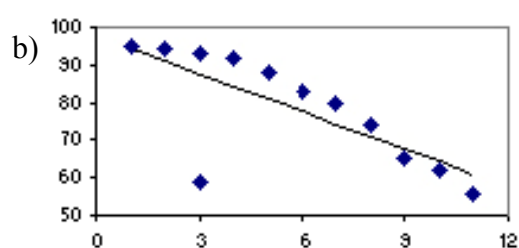
Materials List: Scatterplots and Correlations BLM, computer lab, Regression Line and Correlation BLM, RAFT Writing BLM (teacher only), paper, pencils

One technique for displaying a relationship between two variables is to draw a scatterplot. In this activity, students will explore the correlation between two variables by examining scatterplots and their respective correlation coefficients.

- Take students to a computer lab. If possible, let each student have his or her own computer. If not, allow students to work in small groups of 2-3.
- Hand out the Scatterplots and Correlations BLM.
- Without providing any instruction, ask students to label each scatterplot using the given choices.
- Give students time to compare and adjust answers before discussing the results as a class.
- Let volunteers share their results with the class. Students should justify their answers.
- Hand out the Regression Line and Correlation BLM.
- Tell students to go to the designated web site.
- Tell students to practice plotting points, drawing regression lines, and examining  $r$ -values, using the interactive applet. Walk around to assist students in utilizing the applet.
- Once students feel comfortable with the applet, they should answer questions #1-5.
- Give students time to compare and adjust answers before discussing the results as a class.
- At the end of the Scatterplots and Correlations BLM, students begin to explore the effects of outliers. Outliers warrant further discussion.
- Ask students to define outlier in their own terms.
- Let volunteers share their definitions with the class.  
*Outliers are values or points that have large residuals and diverge from the overall pattern of the rest of the data.*
- Draw or project the following scatterplots on the board. One plot includes an outlier and the other plot does not.



Regression equation:  $\hat{y} = 104.78 - 4.10x$   
Coefficient of determination:  $R^2 = 0.94$

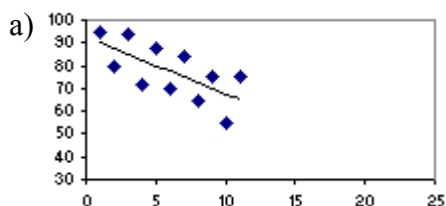


Regression equation:  $\hat{y} = 97.51 - 3.32x$   
Coefficient of determination:  $R^2 = 0.55$

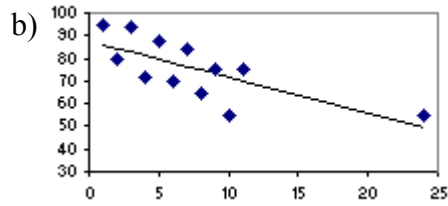
- Ask students which plot includes the outlier.
- Ask students to describe the effect the outlier had on the linear regression equation.
- Let volunteers share their answers with the class.
 

*The outlier changed both the slope and the y-intercept of the linear regression equation. It also dramatically decreased the ability of the explanatory variable to predict the response variable.*
- Ask students if it is possible for an outlier to have no effect on the regression equation.
- Let volunteers share their answers with the class. Volunteers should come to the board and draw an example.
 

*An outlier may not have an effect on the regression equation if it follows the pattern of the rest of the data.*
- Tell students that some outliers are considered influential points. Influential points are data points with extreme values that have a strong effect on the regression equation.
- Draw or project the following scatterplots on the board. One plot includes an influential point and the other plot does not.



Regression equation:  $\hat{y} = 92.54 - 2.5x$   
Coefficient of determination:  $R^2 = 0.46$



Regression equation:  $\hat{y} = 87.59 - 1.6x$   
Coefficient of determination:  $R^2 = 0.52$

- Ask students to compare the linear regression equations and describe the effect of the influential point.
- Let volunteers share their answers with the class.
 

*The influential point had a greater impact on the slope than the outlier in the previous example, and it did not reduce the coefficient of determination. In fact, the coefficient of determination actually rose with the inclusion of the influential point.*
- The existence of an influential point can be determined by comparing the regression equations with and without the point in question. If the difference between the two equations is great, the point is influential. If the difference is slight, the point is not influential.
- While still in the computer lab, let students conduct more research on outliers and influential points as they relate to bivariate data. This will take at least one class period.
- Now that students have had an opportunity to acquire more information about outliers and influential points, introduce *RAFT writing* ([view literacy strategy descriptions](#)).

- *RAFT* is an acronym that stands for: Role of the writer, Audience to whom the *RAFT* is being written, the Form of writing (letter, poem, song, etc.), and Topic (the focus of the writing).
- The purpose of *RAFT writing* is to give students an opportunity to apply and extend their understandings in unique and creative ways.
- To help students understand the process, read the RAFT Writing BLM to the class.
- Ask students if they have any questions about the four components.
- Allow students to help each other by sharing ideas and exchanging papers for additional input.
- Each student should turn in his or her own paper. Since this is the first time students have been exposed to *RAFT writing*, a participation grade should be given instead of a formal grade.
- If time permits, allow volunteers to share their writings with the class. Students should listen for accuracy and logic in their classmates' *RAFTs*.

### Activity 3: The Least Squares Line (GLEs: 20, 22, 29)

Materials List: Least-Squares Line BLM, Hospitals BLM, graphing calculators, paper, pencils

The least-squares line is an example of a regression equation. A regression equation expresses the relationship between the observed values of one variable and the predicted values from the other variable. In this activity, students will learn how to construct the least squares line.

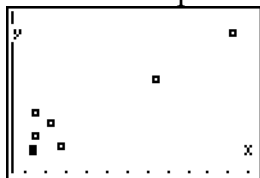
- Allow students to form groups of 3 – 4.
- Hand out graphing calculators and the Least Squares Line BLM.
- Tell students that the Cost of Living Index is based on a national average of 100.
- Ask students to identify the independent and dependent variables. Students should justify their answers.

*The Average Annual Pay is the dependent variable because it is determined by the Cost of Living Index for each city. Therefore, the Cost of Living Index is the independent variable.*
- Ask students if there is a direct or inverse relationship between the two variables. Students should justify their answers.

*Since the Average Annual Pay increases as the Cost of Living Index increases, there is a direct relationship.*
- Ask students to enter the independent variable in List 1 of the graphing calculator, and the dependent variable in List 2.
- Ask students to determine an appropriate window for this data set. Let volunteers share their windows with the class.

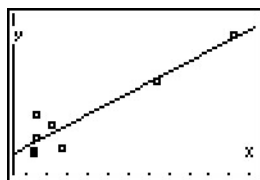
*Windows will vary. A good window allows all data points to be clearly seen.*

Asks students to examine their scatterplots. If possible, allow one student to come to the front of the class and project his or her scatterplot on the board.



- Ask students what type of function could model the data.  
*Linear function*
- Ask volunteers to come to the board and draw a line of best fit for the data. It is helpful if each student draws his or her line with a different colored marker.
- Ask the class which line seems to fit the data the best. Students must justify their answers.  
*Answers will vary*
- Student answers should lead to a discussion of how to mathematically determine the line of best fit (regression line).
- Referring back to #1 of the Least Squares Line BLM, ask students what the vertical line segments represent on the possible candidates for the line of best fit.  
*They represent the vertical distance between each data point and the line of best fit.*
- Ask students how the vertical distances can be used to determine the line of best fit. Students should justify their answers.  
*The line of best fit would be the one with the smallest distances.*
- Ask students how they would differentiate data points lying above the line of best fit from those lying below the line of best fit. Students should justify their answers.  
*Data points lying above the line of best fit have positive vertical distances. Data points lying below the line of best fit have negative vertical distances.*
- Ask students what could happen if they added all of the positive and negative distances.  
*The positive and negative distances could cancel each other out.*
- Ask students why this is a major problem. Students should justify their answers.  
*If they cancel each other out, there will not be a numerical value for comparison purposes.*
- Ask students how they might resolve the problem. Students must justify their answers.  
*Square the distances to eliminate negative values*
- Tell students that the line with the smallest sum of the squares is considered the line of best fit. This particular line of best fit is called the least squares line.
- Based on the vertical distances, ask students which line seems to be the best fit for the data. Students should justify their answers.  
*Line #1 appears to have the smallest vertical distances.*
- Ask students to work in groups of 3 – 4 to complete the table in #2 of the Least Squares Line BLM.
- Tell students to check their results by examining the 2-variable statistics using the graphing calculator.
- Once students have completed and corrected their tables, move on to the formulas for calculating the least squares line (#3 of the Least Squares Line BLM).
- Ask students to calculate  $SS_{xy}$ . Note:  $SS$  stands for sum of squares. The formula for  $SS_{xy}$  provides an efficient way of calculating  $\sum (x - \bar{x})(y - \bar{y})$ .

- Let volunteers share their results with the class.
- Ask students to calculate  $SS_x$ . Note: The formula for  $SS_x$  provides an efficient way of calculating  $\sum (x - \bar{x})$ .
- Let volunteers share their results with the class.
- Ask students to calculate the slope of the least squares line.
- Let volunteers share their results with the class.
- Ask students to calculate the  $y$ -intercept of the least squares line.
- Let volunteers share their results with the class.
- Ask students to write the equation of the least squares line.
- Let volunteers share their results with the class.
- Moving on to #4 of the Least Squares Line BLM, ask students to compare their least squares lines with the calculator's least squares line.
- Let volunteers share their results with the class.  
*Be careful, the calculator's regression equation is  $\hat{y} = ax + b$ , not  $\hat{y} = a + bx$ .*
- Ask students why the two equations are not identical.  
*When calculating the slope and  $y$ -intercept by hand, values were rounded.*
- Tell students to draw the calculator's regression line on the scatterplot.



- Moving on to #5 of the Least Squares Line BLM, ask students what the slope and the  $y$ -intercept represent for this particular data set.
- Let volunteers share their answers with the class.
- Moving on to #6 of the Least Squares Line BLM, ask students to use the least squares line to find the average annual salary for a city with a cost of living index of 100.
- Let volunteers share their results with the class.
- Tell students that using the regression line to predict values for the dependent variable, when values for the independent variable lie between given independent values, is called interpolation.
- Moving on to #7 of the Least Squares Line BLM, ask students to predict the average annual salary for a city with a cost of living index of 80.
- Let volunteers share their results with the class.
- Using the regression line to predict values for the dependent variable, when values for the independent variable lie beyond given independent values, is called extrapolation. Extrapolation should be approached with care because there is no way of knowing if data values outside of the sample range will follow the same pattern.
- Moving on to #8 of the Least Squares Line BLM, ask students to discuss any limitations for the model. Students should justify their answers.
- Let volunteers share their limitations with the class.
- Hand out the Hospitals BLM.
- Give students 20-30 minutes to complete the Hospitals BLM.

- Allow groups to compare answers and make adjustments before discussing the results as a class.

**Extension:** The least squares method is not the only method for determining a regression equation. Another method is called the median-median line. If time permits, or if some students finish early, let them find the median-median line for the Cost of Living and Hospital data sets.

#### Activity 4: The Correlation Coefficient and Coefficient of Determination (GLE: 20)

Materials List: Correlation Coefficient and Coefficient of Determination BLM, graphing calculators, paper, pencils

In this activity, students will learn how to compute the correlation coefficient and the coefficient of determination by hand. Students will also learn how to interpret the values of each in order to determine how well a model fits given data.

- Tell students that in previous activities, we let the graphing calculator find the correlation coefficient ( $r$ ). Now, we will learn how to calculate the correlation coefficient (also known as Pearson's product-moment correlation coefficient) by hand.
- Ask students what the range of values is for the correlation coefficient and what the values represent.
- Let volunteers share their answers with the class.
  - $-1 \leq r \leq 1$  ;  $-1$  represents a perfect negative linear relationship ;  $1$  represents a perfect positive linear relationship ;  $0$  represents no linear relationship
- Write the formula for finding the correlation coefficient on the board.

$$r = \frac{SS_{xy}}{\sqrt{(SS_x)(SS_y)}}$$

- Ask students what they must know in order to use this formula.
- Let volunteers share their answers with the class.

$$SS_{xy} = \sum xy - \frac{(\sum x)(\sum y)}{n} \quad SS_x = \sum x^2 - \frac{(\sum x)^2}{n} \quad SS_y = \sum y^2 - \frac{(\sum y)^2}{n}$$

- Since the range of  $r$  values is small,  $r$  is sensitive to rounding. Therefore, tell students to carry as many decimals as possible until the last step.
- Write the following data set about high school seniors on the board.

<b>IQ</b>	76	88	91	93	99	104	115	120
<b>GPA</b>	1.7	2.5	2.9	2.6	3.5	3.4	2.2	3.8

- Ask students which is the explanatory variable and which is the response variable. Students must justify their answers.
  - IQ is the explanatory variable because it will be used to predict GPA. Thus, GPA is the response variable.*
- Ask students to enter the data into List 1 and List 2 in the calculator.
- Ask students to compute the two-variable statistics by pressing the 2-VAR key.
- Ask students to compute the correlation coefficient.

- Let volunteers come to the board and share their results with the class.

$$SS_{xy} = \sum xy - \frac{(\sum x)(\sum y)}{n} = 2,264 - \frac{(786)(22.6)}{8} = 43.55$$

$$SS_x = \sum x^2 - \frac{(\sum x)^2}{n} = 78,692 - \frac{(786)^2}{8} = 1,467.5$$

$$SS_y = \sum y^2 - \frac{(\sum y)^2}{n} = 67.4 - \frac{(22.6)^2}{8} = 3.555$$

$$r = \frac{SS_{xy}}{\sqrt{(SS_x)(SS_y)}} = \frac{43.55}{\sqrt{(1467.5)(3.555)}} \approx 0.60295$$

- Ask students to compute the regression line in order to compare their  $r$  values with the one generated by the calculator. The answers should be almost exact.
- Ask students to interpret the meaning of the correlation coefficient.
- Let volunteers share their answers with the class.
 

*This  $r$  value indicates a moderate positive linear relationship between the IQ and GPA of the given sample.*
- Ask students what moderate means.
- Let volunteers share their answers with the class.
 

*Moderate is a relative term. It does not say a lot about the strength of the linear relationship. It simply provides a benchmark between no linear relationship and a strong linear relationship.*
- Ask students to go back to their *vocabulary cards* and define coefficient of determination.
- Let volunteers share their definitions with the class.
 

*The coefficient of determination is a number that measures the proportion of variance in the response variable explained by the regression line and explanatory variable ( $0 \leq r^2 \leq 1$ ). In other words, it is the proportion of explained variance to the total variance.*
- Ask students to calculate and interpret the meaning of the coefficient of determination.
- Let volunteers share their answers with the class.
 

*$r^2 \approx 0.36355$  This means that IQ accounts for about 36% of the variance of GPA values. Other factors and random chance account for 64% of the variance.*
- Ask students which measure of correlation, the correlation coefficient or the coefficient of determination, was more meaningful. Students must justify their answers.
 

*Answers will vary. However, most students will prefer the percentage value because it offers more of a benchmark for comparison than the correlation coefficient.*
- It is extremely important for students to understand that the existence of a relationship does NOT imply causality. Both  $r$  and  $r^2$  are measures of relationship. They have nothing to do with cause and effect. True experiments must be conducted in order to infer cause and effect.
- Allow students to form groups of 3-4.
- Hand out the Correlation Coefficient and Coefficient of Determination BLM.
- Give each group about 20 minutes to complete the Correlation Coefficient and Coefficient of Determination BLM.
- Give groups time to compare and adjust answers before discussing the results as a class.

- If students perform well, assign the data set in Sample Assessments for a grade before discussing the results as a class.

### Activity 5: Residual Plots (GLEs: 20, 24)

Materials List: Residual Plots BLM, graphing calculators, paper, pencils

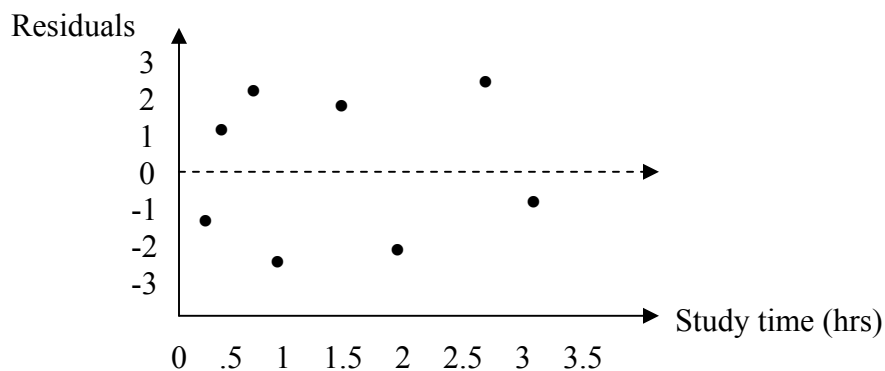
In the previous activity, students learned how to analyze the strength of a linear model for a set of data. They did this numerically by calculating values for  $r$  and  $r^2$ . In this activity, students will learn how to analyze the strength of a linear model for a set of data using graphs. They will create and interpret residual plots by graphing the points  $(x, y - \hat{y})$ . If the residual plot shows a random dispersion of points (i.e. no apparent pattern), the linear regression equation is a good fit for the data.

- Write the following data set on the board.

<b>Study Time (hrs)</b>	.25	.5	.75	1	1.5	2	2.75	3.2
<b>Exam Grade out of 100</b>	54	60	65	62	76	80	96	100

- Tell students to enter the data into List 1 and List 2 of the graphing calculator.
- Tell students to examine the scatterplot.
- Ask students if a linear model would be a good fit for this data.
- Ask students to calculate the linear regression model, correlation coefficient, and the coefficient of determination using the graphing calculator.
- Let volunteers share their results with the class.  

$$\hat{y} = 51.1715092 + 15.5337176x \quad ; \quad r \approx 0.9932 \quad ; \quad r^2 \approx 0.9864$$
- Ask students if the numerical measures of fit agree with the pattern of the scatterplot.  
*Yes: Graphically the data appeared to follow a linear pattern, the correlation coefficient indicates a strong positive linear relationship, and the coefficient of determination states that about 98.64% of the variance in the exam grade can be accounted for by the number of hours spent studying.*
- Allow students to form small groups of 3-4.
- Assign each group one of the eight data points.
- Ask students to define residual and explain how to calculate it.
- Let volunteers share their definitions with the class.  
*A residual is the difference between the observed value and the predicted value. To calculate the residual, subtract the value predicted by the regression model from the observed value  $(y - \hat{y})$ .*
- Ask each group to calculate the residual for its particular point.
- While students are calculating the residuals, draw an x-axis (labeled as study time in hours) and a y-axis (labeled as residuals) on the board.
- Allow one member of each group to come to the board and plot his or her residual.



- Let students compare the scatterplot they drew on the board with the one generated by the graphing calculator.
- Tell students to enter the residual equation into List 3.  

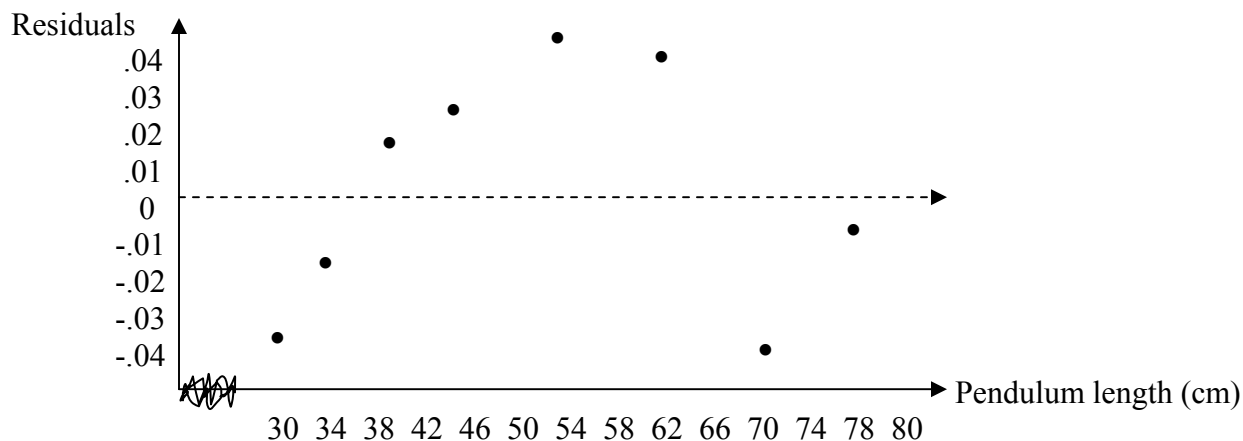
$$\text{residual} = \text{List 2} - (51.1715092 + 15.5337176x \bullet \text{List 1})$$
- Tell students to draw the residual plot (List 1, List 3).
- Ask students if they see a pattern.
- Let volunteers share their thoughts with the class.  
*The points are randomly dispersed. There is no real pattern, but students should notice that half of the points lie above the 0 residual mark and half lie below it.*
- Explain that for a linear model to be a good fit for the data, the points of a residual plot should be randomly dispersed. If a particular pattern or shape is evident, then a linear model is not a good fit and other models should be explored.
- Write the following data set on the board.

<b>Pendulum Length (cm)</b>	30	34	40	46	54	62	70	80
<b>Period (sec)</b>	2.2	2.33	2.53	2.7	2.94	3.15	3.3	3.6

- Ask students what is meant by the period of the pendulum.
- Let volunteers share their answers with the class.  
*The period of a pendulum is the time it takes for the pendulum to make one complete swing (i.e. time it takes to end back where it started).*
- Tell students to enter the data into List 1 and List 2 of the graphing calculator.
- Tell students to examine the scatterplot.
- Ask students if a linear model would be a good fit for this data.
- Ask students to calculate the linear regression model, correlation coefficient, and the coefficient of determination using the graphing calculator.
- Let volunteers share their results with the class.  

$$\hat{y} = 1.40949545 + 0.02758181x ; r \approx 0.99793824 ; r^2 \approx 0.99588074$$
- Ask students if the numerical measures of fit agree with the pattern of the scatterplot.  
*Yes: Graphically the data appeared to follow a linear pattern, the correlation coefficient indicates a strong positive linear relationship, and the coefficient of determination states that about 99.59% of the variance in the exam grade can be accounted for by the number of hours spent studying.*
- Assign each group one of the eight data points.
- Ask each group to calculate the residual for its particular point.

- While students are calculating the residuals, draw an x-axis (labeled as pendulum length in cm) and a y-axis (labeled as residuals) on the board.
- Allow one member of each group to come to the board and plot his or her residual.



- Let students compare the scatterplot they drew on the board with the one generated by the graphing calculator.
- Tell students to enter the residual equation into List 3.  

$$\text{residual} = \text{List 2} - (1.40949545 + 0.02758181 \bullet \text{List 1})$$
- Tell students to draw the residual plot (List 1, List 3).
- Ask students if they see a pattern.
- Let volunteers share their thoughts with the class.  
*Even though half of the points lie above the 0 residual mark and half lie below it, the points form a parabolic pattern.*
- Ask students what the graphical analysis means for the linear regression model.
- Let volunteers share their results with the class.  
*Even though the numerical analysis (correlation coefficient and coefficient of determination) indicated a strong positive linear relationship, the graphical analysis states that a linear model is inappropriate.*
- Tell students to return to the original scatterplot.
- Ask students what type of function might be a better fit for the data. Students must justify their answers.  
*A radical function might be a good fit since the data is concave down.*
- Tell students to state all of the available function models in the graphing calculator.  
*linear, quadratic, cubic, quartic, logarithmic, exponential, power, sine, and logistic*
- Ask students which of the above models would be the best fit for the data. Students must justify their answers both graphically and numerically.
- Let volunteers share their answers with the class.  
*Graphically, the following models can be eliminated because they are not concave down for this data set: quadratic, cubic, quartic, exponential, and sine. Numerically, the correlation coefficients for the remaining models are: Logarithmic ( $r \approx 0.99586461$ ) ; Power ( $r \approx 0.99939417$ ) ; Logistic (ignore this model for now since  $r$  is not given and it is generally used for categorical explanatory variables). Therefore, the power model is the best fit.*
- Ask students to generate the power regression model using the graphing calculator.

- Let volunteers share their results with the class.  
 $y = 0.40548457x^{0.49621646}$
- Ask students if they notice anything about this power model.
- Let volunteers share their observations with the class.  
*The exponent of the power model is close to 0.5 which is the square root function.*
- Hand out the Residual Plots BLM.
- Give each group about 30 minutes to complete the Residual Plots BLM. See Sample Assessments for scoring guidelines.

### Activity 6: Data Transformations (GLEs: 19, 20, 24, 27, 29)

Materials List: Achieving Linearity BLM, graphing calculators, poster board (1 per group), paper, pencils

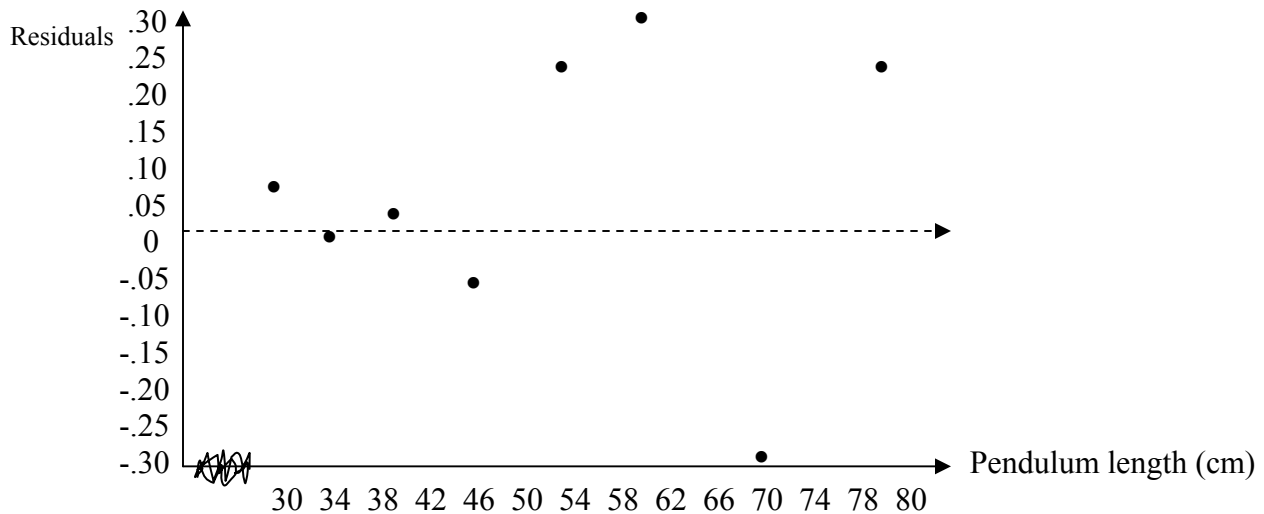
When bivariate data is linearly related, correlation and least squares linear regression are powerful analytical tools. However, what happens when the shape of the raw data and/or the residual plot suggest that a linear model is inappropriate? In this activity, students will attempt to answer that question by achieving linearity through data transformation.

- In Unit 5, students learned how to linearize exponential data by taking the logarithm of both sides of the regression model.
- Write the pendulum data set from Activity 5 on the board.

<b>Pendulum Length (cm)</b>	30	34	40	46	54	62	70	80
<b>Period (sec)</b>	2.2	2.33	2.53	2.7	2.94	3.15	3.3	3.6

- Write the power regression equation found in activity 5 on the board.  
 $y = 0.40548457x^{0.49621646}$
- Tell students to draw the scatterplot using the graphing calculator.
- Ask students how they might linearize this data set.
- Let volunteers share their ideas with the class.  
*Since the power model is very close to a square root function (i.e.  $\frac{1}{2}$  power), squaring the response variable values could linearize the data.*
- Since the regression model is not a perfect square root, ask students if there is another way to linearize the data.
- Let volunteers share their ideas with the class.  
*One way to linearize power functions is to transform the response variable by raising it to the reciprocal power.*
- Tell students to return to List 1 and List 2 in the graphing calculator.
- Tell students to enter (List 2)  $\wedge$  (1/0.49621646) into List 3.
- Tell students to draw and describe the scatterplot (List 1, List 3).  
*The scatterplot is now linear.*
- Tell students that the process of changing raw data to achieve linearity is called data transformation.
- Ask students to find the linear regression for the explanatory variable (List 1) and the transformed data (List 3).

- Let volunteers share their answers with the class.  
 $\hat{y} = -0.0535528 + 0.16324533x$
- Ask students to find and interpret the correlation coefficient and coefficient of determination.
- Let volunteers share their answers with the class.  
 $r \approx 0.99887556$  ;  $r^2 \approx 0.99775239$  Both indicate a strong positive linear relationship between the explanatory variable and the transformed response variable.
- Ask students to enter the residuals into List 4.
- Ask students to draw the residual plot (List 1, List 4) using the graphing calculator.
- Let volunteers come to the board to plot each residual point.



- Ask students what the residual plot says about the transformed data and the raw data.
- Let volunteers share their results with the class.  
*Since the residual plot is randomly dispersed, a linear model is appropriate for the transformed data. Thus, a power regression is appropriate for the raw data. However, since the majority of points lie above the 0 residual mark, the regression equation has a tendency to underestimate the response variable. Despite this flaw, the lack of a residual pattern makes the power regression a better fit than the linear regression.*
- If time permits, allow students to explore other models for the pendulum data. Students should try to balance the numerical and graphical analyses to find a better fit. In other words, they might have to sacrifice a small reduction in the coefficient of determination, in order to find a residual plot with a better dispersion of points above and below the zero residual line.
- Write the following data set on the board.

<b>Number of Workers</b>	1	2	3	4	5	6	7	8
<b>Job Time (hrs)</b>	48	23	16.5	12	9	8	7	5.5

- Ask students what type of relationship is suggested by the raw data.
- Let volunteers share their ideas with the class.  
*Since the number of hours decreases as the number of workers increases, the relationship is an inverse or indirect one.*

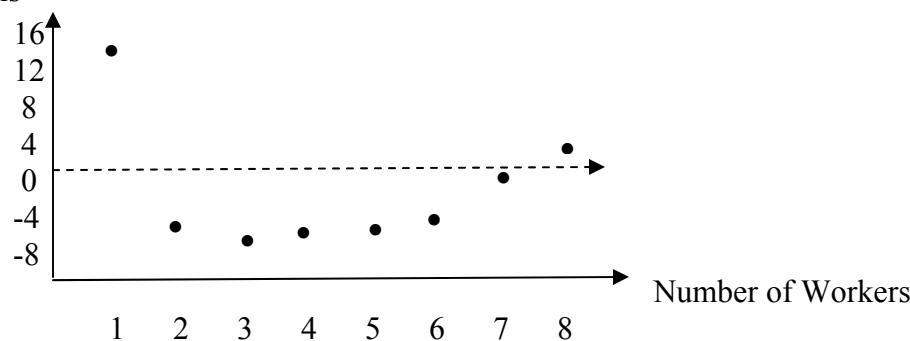
- Tell students to examine the scatterplot using the graphing calculator.
- Ask students if a linear model would be a good fit for this data.
- Let volunteers share their results with the class.
 

*No: The raw data follows a curvilinear pattern.*
- Ask students to calculate the linear regression model, correlation coefficient, and the coefficient of determination using the graphing calculator.
- Let volunteers share their results with the class.
 

$\hat{y} = 37.875 - 4.8\bar{3}x$  ;  $r \approx -0.8391457$  ;  $r^2 \approx 0.70416554$
- Ask students if the numerical measures of fit agree with the pattern of the scatterplot.
 

*No: The scatterplot displays a non-linear pattern, but the correlation coefficient and the coefficient of determination both suggest a moderate to strong negative linear relationship.*
- Ask students to enter the residuals into List 3 and draw the residual plot (List 1, List 3).
- Ask students what the residual plot says about the raw data.
- Let volunteers share their answers with the class.

Residuals



*Since the residual plot is not randomly dispersed, a linear model is inappropriate for the raw data.*

- Ask students what non-linear models might fit the data.
- Let volunteers share their ideas with the class.
 

*Since the raw data is concave up, possible models include: quadratic, cubic, quartic, exponential, power, and sine. However, if students remember the inverse function from Algebra II, they should consider the power function to be a good choice.*
- Ask students to examine the correlation coefficient and coefficient of determination for each of the possible models.
- Ask students which model appears to fit the raw data the best.
- Let volunteers share their ideas with the class.
 

*The power regression model has the best correlation coefficient and coefficient of determination.*  $\hat{y} \approx 48.0842515x^{-1.0147811}$  ;  $r \approx -0.9983753$  ;  $r^2 \approx 0.99675336$
- Ask students what transformation could be used to linearize the data for graphical analysis.
- Let volunteers share their ideas with the class.
 

*Raising the response variable to the reciprocal power should linearize the data.*
- Ask students to enter the transformed data (List 2  $\wedge$  1/-1.0147811) into List 3.

- Ask students to examine and describe the scatterplot for the transformed data (List 1, List 3).

*The transformed data follows a linear pattern.*

- Ask students to write the linear regression equation for the transformed data.

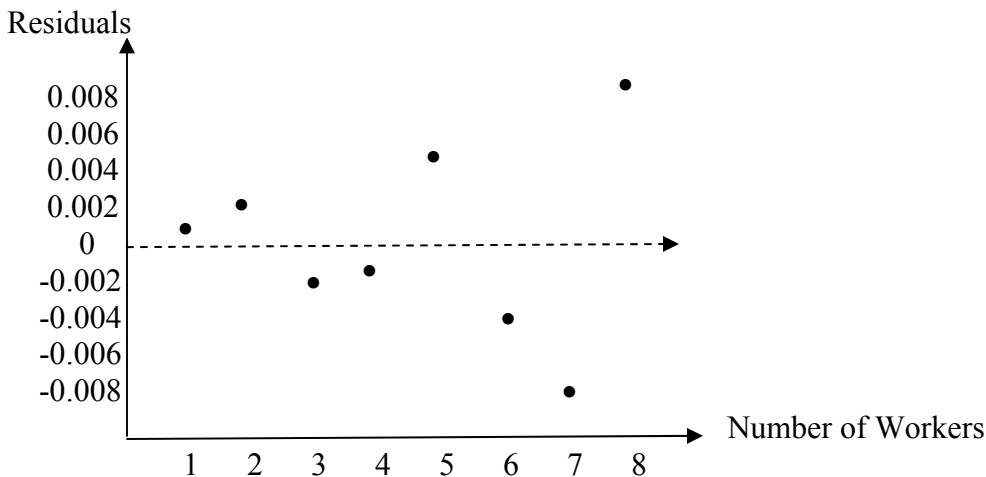
$$\hat{y} \approx -0.001632 + 0.02241857x$$

- Ask students to find the correlation coefficient and coefficient of determination.
- Let volunteers share their answers with the class.

$$r \approx 0.99541486 ; r^2 \approx 0.99085075$$

- Ask students if the numerical measures of fit agree with the pattern of the scatterplot.  
*Yes: The scatterplot displays a linear pattern, and the correlation coefficient and coefficient of determination suggest a strong positive linear relationship.*

- Ask students to enter the residuals for the transformed data into List 4.
- Ask students to draw and interpret the residual plot (List 1, List 4) using the graphing calculator.



*The residual plot displays a random dispersion of points, so the linearization was successful. Therefore, the power regression is a good fit for the raw data.*

- To summarize data transformations and linearization, write the following table on the board.

Model	Transformation
Quadratic Function	
Square Root	
Reciprocal or Inverse Function	
Power Function	
Exponential Function	
Logarithmic Function	

- Ask students to complete the table.

- Let volunteers share their answers with the class.

Model	Transformation
Quadratic Function	$\sqrt{y}$
Square Root	$y^2$
Reciprocal or Inverse Function	$\frac{1}{y}$
Power Function	$y \wedge \text{reciprocal power}$ OR $\log x \text{ and } \log y$
Exponential Function	$\log y$
Logarithmic Function	$\log x$

- Remind students that numerical analyses (correlation coefficient and coefficient of determination), when used in isolation, can be misleading. Recall the pendulum data. The correlation coefficient and the coefficient of determination both indicated a strong positive linear relationship even though the scatterplot clearly showed a curvilinear pattern. Thus, it is important to conduct graphical analyses (scatterplots and residual plots for both raw and transformed data).
- Hand out the Achieving Linearity BLM.
- Allow students to form groups of 3-4.
- Give students 2 - 3 days to complete the Achieving Linearity project.
- If possible, take an additional day to let group members share their data and results with the class.
- See Sample Assessments for scoring guidelines.

Extension: If time permits, or if students finish early, introduce the  $\log x$  and  $\log y$  transformation for power regressions.

### Sample Assessments

#### General Assessments

- Each student will create an eighth entry for his or her *learning log* ([view literacy strategy descriptions](#)). Remember, students will be adding entries to the *learning logs* throughout the school year. The eighth entry will have the title, “Bivariate Data.” This assessment piece can be formal or informal in nature. Informally, students would be assigned a grade for writing the entry. This will let you know what students do and do not understand about univariate statistics. Formally, students can be graded based on the inclusion of graphical analyses (scatterplots & residual plots) and numerical analyses (correlation coefficient and coefficient of determination) of linear and non-linear relationships.

- Each student will continue to add terms from this unit to the glossary started in Unit 1. This glossary will be included in a student portfolio that will be graded at the end of each semester.
- Each student will complete a “spiral” comprised of 5 -10 teacher-made problems. These problems should center on the topics covered in Units 1, 2, 3, 4, 5, 6, 7, and 8. Again, at least half of the problems should be in multiple choice form. This will help prepare students for the unit test and the ACT/SAT exams. “Spirals” should be assigned every 2-3 weeks to ensure that students understand and retain important concepts and procedures.

### Activity-Specific Assessments

- Activity 3: Least Squares Line

Each student will turn in his or her own Hospitals BLM. For #2, assign 5 points for the correct slope and another 5 points for the correct y-intercept. For #3, assign 2 points for the correct interpretation of the slope, 2 points for the correct interpretation of the y-intercept, and 1 point for stating that the y-intercept does not make sense for the data set. For #4, assign 5 points for the correct regression equation. For #5, assign 5 points for the correct prediction. Note that both the truncated and rounded values are listed. Accept either one or decide as a class how to handle parts of discrete objects. For #6, assign 5 points for acceptable limitations. The point total for the Hospitals BLM is 25.

- Activity 5: Residual Plots

Each student will turn in his or her own Residual Plots BLM.

#1: assign 2 points for the correct answer and 3 points for the correct explanation

#2: assign 2 points for the correct answer and 3 points for the correct explanation

#3: assign 3 points for the correct regression equation

assign 1 point for the correct interpretation of the slope

assign 1 point for the correct interpretation of the y-intercept

#4: assign 1 point for the correct correlation coefficient

assign 1 point for the correct coefficient of determination

assign 1 point for the correct interpretation of the correlation coefficient

assign 2 points for the correct interpretation of the coefficient of determination

#5: assign 5 points for the correct residual plot

#6: assign 5 points for a correct explanation.

The point total for the Residual Plots BLM is 30.

➤ Activity 6: Achieving Linearity

Each group will turn in one Achieving Linearity project. Award 5 points for each correct response to #1-14. This makes the project total 70 points. Additional points can be awarded for creativity and, if time permits, an oral presentation.